MALTHUS' PRODUCTION FUNCTION

by

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In commenting on Malthus' geometric progression for population and arithmetic progression for food, G.J. Stigler stated (1952, p. 190):

I Malthus' production function is: $L = 2^{P-1}$ where L is labour (proportional to population) and P is produce;

II if wages equal the marginal product of labour, then wages = $dP/dL = 1/(L.\ln 2)$; and the aggregate wages bill = $L.dP/dL = 1/\ln 2$, which is a constant and is therefore independent of the size of the labour force;

III and 'population simply could not grow!'

A recent note by B.L. Boulier and J.W. Wilson (1987) has disputed point III. The purpose of the present note is to question Stigler's points I and II, and to comment on some of the arguments of Boulier and Wilson: and then to present a generalised model of Malthus' position on population and food.

I THE PRODUCTION FUNCTION

Stigler compared the two series given in Malthus 1966, p. 25:

Population (L)	1	2	4	8	(1)
Produce (P)	1	2	3	4	(2)

But Stigler did not refer to the two series given (in words, not numerals) a little earlier on pp. 23-4 of the *Essay*:

'The population of the Island is computed to be about seven millions; and we will suppose the present produce equal to the support of such a number. In the first

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twenty-five years the population would be fourteen millions; and the food being also doubled, the means of subsistence would be equal to this increase. In the next twenty-five years the population would be twenty-eight millions; and the means of subsistence only equal to the support of twenty-one millions.'

i.e.	Population (L)	7	14	28	56	(3)
	Produce (P)	7	14	21	28	(4)

Which pair of series correctly represents Malthus' views? We submit that Stigler used the wrong series, and that the two series (3) and (4), referring as they do to Malthus' estimate of the population of Britain at the time, are the correct pair. Series (1) and (2) were presented by Malthus merely to illustrate for the non-mathematical reader the difference between a geometric progression and an arithmetic progression, and were not intended to represent the actual state[1] (and the potential growth) of the population and the food supply.

The choice of series has implications for the production function.[2] From series (1) and (2) Stigler concluded that Malthus' production function was

$$L = 2^{P-1} \qquad (5)$$

which can be expressed as

$$P = \frac{\ln L}{\ln 2} + 1 \qquad (6)$$

But the production function linking L and P in (3) and (4) is:

$$L = 7.2^{(P/7)-1} \qquad (7)$$

$$\text{or, } P = \frac{7}{\ln 2}\,[\ln L + \ln 2 - \ln 7] \qquad (8)$$

More generally, if 'a' represents the starting point of both the produce series and the population series[3], then

$$P = \frac{a}{\ln 2}\,[\ln L + \ln 2 - \ln a] \qquad (9)$$

$$= \frac{a}{\ln 2}\,.\ln\left(\frac{2L}{a}\right) \qquad (9a)$$

$$\text{and } L = a.2^{(P/a)-1} \qquad (10)$$

Equations (9), (9a), and (10) represent the production function implicit in Malthus' arithmetic and geometric ratios.

When a = 1, equation (9) becomes identical to Stigler's production function (6), and equation (10) becomes identical to Stigler's production function (5). Stigler's production functions (5) and (6) are thus only special cases of production functions (9) and (10). And Stigler's production functions (5) and (6) cannot be taken to represent the relationship which exists between L and P in series (3) and (4) where, for example, when P = 28, L = 56, not 2^{27}.

II THE MARGINAL PRODUCT OF LABOUR AND THE WAGES BILL

On the basis of Stigler's production function (6), the marginal product of labour $= \dfrac{dP}{dL} = \dfrac{1}{L.\ln 2}$ (11)

But if the correct production function is that given in equation (9) then the marginal product of labour $= \dfrac{a}{L.\ln 2}$ (12)
which is the same as Stigler's version only in the special case where a = 1.

Stigler calculates the wages bill as follows:

Wages bill=(marginal product of labour)x(no. of labourers) (13)

$$= \frac{dP}{dL} . L = \frac{1}{\ln 2} \qquad (14)$$

and concludes that the wages bill is a constant, and is independent of the size of the labour force. If we accept[4] Stigler's method of calculating the wages bill, as expressed in equation (13), but if we employ equation (12) instead of Stigler's equation (11) to calculate the marginal product of labour, then

$$\text{Wages bill} = \frac{a}{\ln 2} \qquad (15)$$

$$= \frac{7}{\ln 2} \quad \text{when } a = 7 \qquad (15a)$$

Thus, using Stigler's method of calculating the wages bill, it is correct to say with Stigler that, given the size of the initial population, the wages bill is constant, but it is not correct to say with Stigler that the wages bill is independent of the size of the labour force. In equation (15), the wages bill is a function of the initial population (a), and hence is a function of the existing labour force.

III POPULATION GROWTH

Having argued that the wages bill could not grow, Stigler concluded (1952, p. 190) that 'population simply could not grow!' Boulier and Wilson (1987, p.95) have criticised that statement on the grounds that the constancy of the aggregate wages bill does not mean that population could not grow; it 'merely indicates that the population could not grow infinitely large, assuming that there is a minimum level of subsistence' (p. 95).

Stigler's conclusion that population could not grow is, to say the least, paradoxical, and could be criticised on further grounds:

(i) Malthus' population series clearly indicates that, over the first 25-year period, population does in fact grow - from 1 to 2 in series (1), and from 7 to 14 in series (3) - without any fall in produce per head. Over subsequent 25 year periods, it was obviously Malthus' intention to argue, not that population could not grow, but that population was limited by the food supply, and hence that population could grow at the same rate as produce[5], i.e. 1, 2, 3, ...; or 7, 14, 21, ...

(ii) In the second and subsequent terms of Stigler's produce series (2), there is a surplus of total produce over the wages bill. This surplus (= P - $1/ln$ 2) will continue to increase (though at a decreasing rate) as the series unfolds into the future, and will be available for the support of population. Stigler says nothing about how this surplus is allocated, and nothing about its effect on population growth. If the wages bill is constant, the *population supported by the wages bill* cannot grow beyond the level at which the average wage falls to the minimum level of subsistence - as explained in Malthus 1966, pp. 29-30. But the *population supported by the increasing surplus* (P - $1/ln$ 2) can continue to grow. It cannot be correct therefore to say that the constancy of the wages bill - as expressed by Stigler in equation (14) - proves that *population as a whole* cannot grow.

IV MAXIMUM INCOME PER HEAD

Boulier and Wilson (1977, p. 95) argue, using Stigler's production function (6), that income per head ($Y = P/L$) is a maximum when the population is 1.36. At that level of population, the maximum level of income per head is 1.06. And the level of production at which Y is a maximum is 1.44. [6]

But if instead of Stigler's version (6) of the production function, we adopt the alternative version (9), then

$$Y = \frac{a}{L\,ln\,2}\,[ln\,L + ln\,2 - ln\,a] \qquad (16)$$

and

$$\frac{dY}{dL} = \frac{a}{L^2\,.\,ln\,2}\,[1 - ln\,2 - ln\,L + ln\,a] \qquad (17)$$

and Y is a maximum when

$$ln\,L = 1 + ln\,a - ln\,2 \qquad (18)$$

$$\text{or } L = \frac{ae}{2} \qquad (19)$$

Thus using Malthus' series (3) and (4) where $a = 7$, maximum income per head occurs where $L = 9.52$ (if $e = 2.72$).

Substituting for L in equation (16), the maximum value of Y

$$= \frac{a}{\frac{ae}{2}\,.\,ln\,2}\,[ln\,\frac{ae}{2} + ln\,2 - ln\,a] \qquad (20)$$

$$= \frac{2}{e.\,ln\,2}$$

$$= 1.06 \quad \text{if } e = 2.72$$

Thus the maximum[7] level of Y is the same for series (3) and (4) as for series (1) and (2), i.e. the maximum level of Y is independent of the size of the initial population. But the level of L at which Y is a maximum is not independent of the size (a) of the initial population.

Having established that maximum income per head occurs at $L = 1.36$, Boulier and Wilson (1977) facetiously added: 'Thus, per capita income presumably reached its peak in the Garden of Eden' (p. 95). But of course that would be true only if Malthus were discussing a population series in which the first term was one person, whereas in fact Malthus in 1798 put the first term at 7 million, and in 1826 at 11 million. This 'Garden of Eden' comment shows that Boulier and Wilson, like Stigler, did not distinguish between the illustrative ratios of series (1) and (2), and the actual magnitudes of series (3) and (4). However, Boulier and Wilson then proceeded to make an adjustment to convert series (1) and (2) to actual magnitudes:

> More formally, since units in our formulation of the production function are undefined, the right hand side should be multiplied by a constant (π) to adjust for units of measurement. (1977, p. 95).

This is essentially the conclusion reached above. But a change from the illustrative units of series (1) and (2) to the actual magnitudes of series (3) and (4) involves a rejection of Stigler's version (6) of Malthus' production function.

V A GENERALISED MODEL

Production function

The foregoing discussion can be generalised as follows:

If population (l) is growing at an instantaneous rate r per annum, population at time t (in years) is

$$L = L(0).e^{rt} \qquad (21)$$

where L(0) = population at initial time t = 0

If production per year increases linearly, production at time t is

$$P = P(0) + bt \qquad (22)$$

where P(0) = production in year 0 which we define as the year ending at time t = 0
b = annual increase in production

$$\text{From (21), } t = \frac{1}{r}\,[\ln L - \ln L(0)] \qquad (23)$$

Substituting for t in (22), we derive the implicit production function

$$P = P(0) + \frac{b}{r} [ln\, L - ln\, L(0)] \qquad (24)$$

If T = length of time in years between observations in the population series (Malthus assumed T = 25)

and R = rate of growth of population between successive observations (Malthus assumed R = 1)

$$\text{then } R = \frac{L\,(0)\, e^{r(t + T)} - L(0)\, e^{rt}}{L(0)\, e^{rt}} \qquad (25)$$

$$= e^{rT} - 1 \qquad (25a)$$

$$\text{and } r = \frac{1}{T}\, ln\, (1 + R) \qquad (26)$$

The relationship between the base-period production P(0) and the base-period population L(0) can be defined as

$$P(0) = g\, L(0) \qquad (27)$$

(Malthus assumed g = 1; see footnote 3.)

Substituting equations (26) and (27) in equation (24), we derive the generalised production function:

$$P = g\, L(0) + bt \,.\, \frac{ln\, L - ln\, L(0)}{ln\, (1 + R)} \qquad (28)$$

In the case of Malthus' series (1) and (2), where g = 1, L(0) = P(0) = 1, T = 25, b = 1/25, and R = 1, production function (28) reduces to the Stigler function (6).

In the case of Malthus' series (3) and (4), where g = 1, L(0) = P(0) = 7, T = 25, b = 7/25, and R = 1, production function (28) reduces to our production function (8).

Marginal product of labour and wages bill

The generalised expression for the marginal product of labour is - from equation (28) -

$$\frac{dP}{dL} = \frac{bT}{L \,.\, ln\, (1 + R)} \qquad (29)$$

and the generalised expression for the wages bill is:

$$\text{wages bill} = \frac{bT}{ln\ (1 + R)} \qquad (30)$$

In the case of series (1) and (2), i.e. when b = 1/25, T =25, R = 1, these general expressions for dP/dL and the wages bill reduce to Stigler's equations (11) and (14). In the case of series (3) and (4), i.e. when b = a/25, T = 25, R =1, they reduce to our equations (12) and (15).

Maximum income per head

The generalised expression for income per head is - from equation (28) -

$$Y = \frac{P}{L} = \frac{g\ L(0)}{L} + \frac{bT ln\ L}{L.ln\ (1+R)} - \frac{bT ln\ L(0)}{L.ln\ (1+R)} \qquad (31)$$

or alternatively, from equations (21) and (22)

$$Y = \frac{P}{L} = \frac{P(0) + bT}{L(0)\ e^{rt}} \qquad (32)$$

Differentiating equation (31) with respect to L, Y is maximised when;

$$\frac{dY}{dL} = - g\ L(0) ln\ (1+R) + bT - bT ln\ L + bT ln\ L(0) = 0 \qquad (33)$$

$$\text{since} \quad \frac{d^2Y}{dL^2} = - \frac{bT}{L} < 0 \qquad (34)$$

In the case of series (1) and (2), where g = 1, L(0) = 1, R = 1, b = 1/25, and T = 25, equation (33) reduces to

$$- ln\ 2 + 1 - ln\ L = 0 \qquad (35)$$

$$\text{and} \quad L = \frac{e}{2} = 1.36 \quad \text{if } e = 2.72$$

In the case of series (3) and (4), where g =1, L(0) = 7, R = 1, b = 7/25, and T = 25, equation (33) reduces to

$$-7\ ln\ 2 + 7 - 7\ ln\ L = 0 \qquad (36)$$

and $L = \frac{e}{2} = 1.36$ if $e = 2.72$

Thus the generalised expression for income per head confirms, as stated above, that the maximum level of income per head is the same for series (3) and (4) as for series (1) and (2), and is independent of the size of the initial population.

From equation (33), maximum Y occurs when

$$bT\,[\ln L - \ln L(0)] = bT - g\,L(0)\,\ln(1 + R) \quad (37)$$

Substituting the left side of equation (37) into equation (28), it follows that maximum Y occurs when

$$P = g\,L(0) + \frac{bT - g\,L(0)\,\ln(1 + R)}{\ln(1 + R)} \quad (38)$$

In the case of series (1) and (2) equation (38) reduces to

$$P = \frac{1}{\ln 2} = 1.44$$

In the case of series (3) and (4) equation (38) reduces to

$$P = \frac{7}{\ln 2} = 10.08$$

The level of t at which Y is a maximum can also be expressed in a generalised form. From equation (32)

$$\frac{dY}{dt} = \frac{e^{-rt}}{L(0)}\,[b - P(0)r - brt] \quad (39)$$

But $P(0) = L(0)$ and $b = \frac{P(0)}{T}$ (see footnote 9)

$$\text{Thus} \quad \frac{dY}{dt} = \frac{e^{-rt}}{T}\,[1 - rt - rT] \quad (40)$$

$$\text{and} \quad \frac{d^2Y}{dt^2} = \frac{-re^{-rt}}{T}\,[1 - rt - rT] \quad (41)$$

Thus Y is a maximum when

$$1 - rt - rT = 0 \tag{42}$$

or $$t = \frac{1}{r} - T$$

or $$t = \frac{T}{ln\,(1 + R)} - T \qquad \text{since } r = \frac{ln\,(1 + R)}{T}$$

Thus if R = 1 and T = 25, the maximum level of Y occurs when t = 11.067 years.

FOOTNOTES

1 In the second (1803) and later editions of the *Essay* the starting point of series (3) and (4) was altered to eleven million.

2 It should be noted that the production function under discussion here refers to a situation which Malthus hoped would not occur, viz. a situation in which population tended to increase more rapidly than produce. It is therefore the production function of his warning or pathological mode, not of his preferred or physiological model.

3 The following statement indicates that Malthus intended his two series to have the same starting point: 'The population of the Island is computed to be about seven millions; and we will suppose the present produce equal to the support of such a number.' (Malthus 1966, p.23)

4 Stigler's method of calculating the wages bill creates a serious anomaly: the wages bill so calculated (viz. $1/ln\,2 = 1.44$) is greater, in the first term of the produce series, than the total produce! The same anomaly occurs using equation (15) when a = 7.

5 Malthus argued that the growth of produce should precede the growth of population, i.e. that *if* produce grows at 1, 2, 3 ..., *then* population can grow at 1, 2, 3 ... .

6 Using Stigler's production function (6),

$$\text{Income per head} = Y = \frac{P}{L} = \frac{1}{L}\left[\frac{ln\ L}{ln\ 2} + 1\right]$$

$$\frac{dY}{dL} = \frac{1}{L^2 . ln\ 2}\left[1 - ln\ L - ln\ 2\right]$$

and Y is at a maximum when $L = \frac{e}{2} = 1.36$ (if $e = 2.72$)

$$\frac{d^2Y}{dL^2} = \frac{1}{L^3 . ln\ 2}\left[ln\ L + ln\ 2 - 2\right]$$

and $\frac{d^2Y}{dL^2} < 0$ when $L = 1.36$

The maximum value of $Y = \frac{1}{1.36}\left[\frac{ln\ 1.36}{ln\ 2} + 1\right]$

$$= 1.06$$

The discrete series (1) and (2) show that produce per head is at a maximum level of unity when L = 1 or L = 2, but the assumption of continuity in the production function situates the point of maximum produce per head at a level of 1.06 when L = 1.36.

The level of P at which Y is a maximum is P = Y.L = (1.06)(1.36) = 1.44. Or alternatively, from equation (5),

$$Y = \frac{P}{L} = \frac{P}{2^{P-1}}$$

$$ln\ Y = ln\ P - (P - 1)\ ln\ 2$$

Differentiating with respect to P:

$$\frac{dY}{dP} = 2^{1-P}(1 - P\ ln\ 2)$$

$$\frac{d^2Y}{dP^2} = \frac{-ln\ 2}{L}$$
$$< 0$$

Thus Y is a maximum when $P = \dfrac{1}{ln\ 2} = 1.44$

7 From equation (17), $\dfrac{d^2Y}{dL^2} = \dfrac{-a}{L^3.ln\ 2}\ [2 - ln\ 2 - ln\ L + ln\ a]$

and $\dfrac{d^2Y}{dL^2} < 0$ when a = 7.

8 A.M.C. Waterman (1987, p.260) derives Malthus' production function in the form

$$F = p + L(ln\ N - q)$$

which when translated into the terminology used in this note becomes our equation (9)

$$P = \frac{a}{ln\ 2}\ (ln\ L + ln\ 2 - ln\ a)$$

However, Waterman reduces his Malthus production function to

$$F = L\ ln\ N$$

which, when translated into our terminology becomes

$$P = \frac{a}{ln\ 2}\ ln\ L$$

Waterman states that his reduced-form equation is 'the equation that summarizes Malthus's 'ratios''. If this were so, then the values of P generated by that equation should be those which appear in Malthus' production series (2) or (4). But the values of P generated by Waterman's reduced-form equations are:

If a = 1, and L = 1 2 3 4

then P = 0 1 2 3 (A)

If a = 7, and L = 7 14 28 56

$$\text{then } P = \frac{7\ ln\ 7}{ln\ 2} \quad \frac{7\ ln\ 14}{ln\ 2} \quad \frac{7\ ln\ 28}{ln\ 2} \quad \frac{7\ ln\ 56}{ln\ 2}$$

or P = 19.65 26.65 33.65 40.65 (B)

and these are not the values of P that appear in Malthus' original series (2) or (4). The terms of series (A) exceed those of series (2) by 1. The terms of series (B) exceed those of series (4) by 12.65. The difference between our production function (9) and Waterman's reduced-form function is: $a - \frac{a\ ln\ a}{ln\ 2}$, which equals 1 when a = 1, and equals 12.65 when a = 7.

Waterman's reduced-form equation also gives a different result for maximum income per head of population. As shown in equation (20), if production function (9) is adopted, then the maximum level of Y (income per head) occurs when $L = \frac{ae}{2}$, and is equal to $\frac{2}{e\ ln\ 2}$. But using Waterman's reduced-form production function,

$$Y = \frac{P}{L} = \frac{a\ ln\ L}{L\ ln\ 2}$$

and the maximum level of Y occurs when L = e and is equal to $\frac{a}{e\ ln\ 2}$. This is obviously unacceptable, because it means that the optimum level of population will always be approximately 2.72 irrespective of whether the initial population is 1, 7, one million, seven million, or 1000 million.

9 In Malthus' production series the annual increase in production equals one-twentyfifth of the initial level of production - 'Let us then take this for our rule, though certainly far beyond the truth; and allow that by great exertion, the whole produce of the Island might be increased every twenty-five years, by a quantity of subsistence equal to what it at present produces.' (Malthus 1966, p.22)

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