The Mathematics of Kalecki's Early Macrodynamics: A Research Note

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Introduction

Almost sixty years have passed since Michal Kalecki presented his initial model of the business cycle. Like his contemporaries Frisch (1933) and Tinbergen (1934), Kalecki's goal is to build a formal model capable of explaining major elements of the business cycle. Clearly he thought his work said something meaningful about the dynamics of capitalist economies, for when it was observed that his model gives rise to paradoxical results, e.g., profitable investments produce prosperity but also sow the seeds of future economic crisis, he writes, "But it is not the theory which is paradoxical but its subject - the capitalist economy." (Kalecki 1937, p. 96).

The purpose of this note is to summarize the results of a study[1] that presents a careful analysis of the mathematical structure, derivation, and interpretation of Kalecki's model published in *Econometrica* (Vol. 3, 1935, pp. 327-344). His 1935 presentation is chosen for two reasons. One, it provides the point of departure for nearly all Kalecki's subsequent work on business cycles and, two, our methodological critique applies to all formal variations of Kalecki's theory of business cycles.[2]

To render his model tractable, Kalecki makes certain economic and mathematical assumptions. The former include assuming that the economy is closed, exhibits no growth, has no government sector, and, a key distributional assumption, all saving is done by the capitalists; workers save nothing. Finally, there are no prices or interest rates in the model. Such radical simplification is often necessary if the analyst is to make any headway in modelling complex phenomenon. As Kalecki [quoted in Feiwel, 1975, p. 159] observes,"... to approach the dynamic process in all its complexity is certainly a hopeless task." We agree. Indeed, we offer no criticism of the economic structure or the simplifying economic assumptions made by Kalecki to derive that structure.

In addition to the simplifying economic assumptions, Kalecki implicitly makes two crucial mathematical assumptions. One, he assumes that a meaningful solution to his fundamental equation can be found by exploring only the first of an infinite series of exponential solutions. Two, he assumes that each of his model's five basic functions is differentiable with respect to time.

A Summary of the Study

This research note summarizes the considerations that lead us to this critique, the questions we address, and the conclusions we reach. Our interest in Kalecki's modelling has been stimulated by two considerations. One, we have become increasingly suspicious of the mathematical derivation by which Kalecki reduces the system of equations in his model to a single difference-differential equation. Clearly something is lost in the process. Kalecki's fundamental equation does not properly retain all of the underlying economic information of the model. Two, modern numerical methods allow us to explore fully the properties of Kalecki's model.[3] Because of the solution techniques available to Kalecki in the early 1930s, he necessarily confines his own use of the model to calculating only the period and damping factor of the business cycle.

Our full paper retraces Kalecki's reduction of his model to one equation and reviews his mathematical treatment of that "fundamental equation". Both aspects of his analysis raise important questions. Using numerical methods and Kalecki's parameter values, we solve Kalecki's model and compare and contrast his results with our more complete solution.[4]

Kalecki seeks exponential solutions to his equation, having the form e^{at}. He finds that infinitely many such solutions exist and argues that realistic values of the underlying parameters lead him to conclude that none of the admissible values for a are real. Kalecki proceeds, without proper mathematical justification, to discard all the exponential solutions except one; he keeps the one with the smallest imaginary part using the (merely) plausible argument that it would provide the dominant period for the business cycle. Given Kalecki's training in engineering, perhaps he had in mind the transmission of sound waves wherein higher frequencies only provide overtones.

Furthermore, as is evident, one of the underlying parameters of Kalecki's model does not survive his derivation of the fundamental equation. The missing parameter is autonomous consumption, c_1, from the linear consumption function, $c(t) = c_1 + \lambda Y(t)$. In his derivation, Kalecki assumes not only continuity of his central economic variable, orders for investment goods, but differentiability of the investment decision function with respect to time. After substituting and combining terms, Kalecki's differentiation of the investment function, with respect to time, results in the loss of c_1. Also lost in the derivation is the capital stock, $K(0)$, at $t = 0$, when the model opens a window on the economy. Both missing elements influence the model's cycle.

The discarding of exponential solutions raises several significant questions:

(1) Can every solution of Kalecki's equation be constructed as an infinite series of the exponential solutions found by Kalecki's procedure? In particular, can the one which represents reality in Kalecki's world be so constructed?

(2) Even if the answer to (1) is yes, with what justification can all the terms in the series, except the first, be discarded?

(3) How does one go about selecting the correct solution from infinitely many? What are the appropriate initial conditions for this mathematical problem that guarantee the existence of a unique solution? The steps in Kalecki's derivation whereby c_1 and $K(0)$ drop out raises additional questions:

(4) Is the future behaviour of the economy independent of the level c_1 and $K(0)$? If so, why?

(5) Is it safe to assume with Kalecki differentiability of the investment decision function?

Kalecki assumes that the underlying parameters of the model are constant across markets and through time. Unfortunately, he is not explicit about the time intervals involved in the analysis. (6) Did he mean that the parameters would be and had been constant forever? Or did Kalecki mean that the parameters are constant only for the window of time during which the economy is analyzed, (say) for the last 50 years? Exactly what is the minimal set of assumptions under which his derivation holds? We conjecture that Kalecki did not even think about these questions. However, the questions are far from trivial.

Briefly, we find that: The answers to the twin questions in (1) are yes and yes. This is all the more remarkable because a rigorous treatment leading to a careful understanding of the questions in (1) was not available until the mid-20th century. The book by Bellman and Cooke (1963) is the first general reference available that addresses these issues. As for (2) we find that the intuition that leads Kalecki to consider only one of these exponential solutions is dependable with respect to the period and the damping factor; however, these are the only features shared by the correct solution and Kalecki's one term solution (e.g., cycle phases and amplitudes differ).

Moreover, the calculated period and damping factor are extremely sensitive to the values of the model's parameters. A small change in the value of any one parameter leads the model to collapse. This observation was made first by Frisch and Holme (1935). Evidently Kalecki chooses a combination of parameter values such that the model would exhibit zero damping.

One must be very careful in answering (3). A superficial response is that the appropriate initial conditions for Kalecki's equation is to specify the past history of the solution for $-\theta \leq t \leq 0$; where θ is Kalecki's investment order/delivery lag. This is equivalent to specifying the past history for a interval equal to one lag period. In one sense, this is correct. However, the problem is that the "actual history" of Kalecki's function is not available. And if one supplies a hypothetical history of the function that turns out to be spurious, i.e., a history that does not obey the Kaleckian investment decision function, one obtains a fallacious future time path for investment. A detailed discussion of this problem is given.

Likewise, the answer to (4) is subtle. Yes, the future behavior of the economy depends on c_1 and K(0). Understanding the trouble caused by the loss of these terms leads to the realization that the answer to (6) is essentially that Kalecki's derivation required that his model has held for all past time, with no changes in any parameters. Regarding (5), we note that the assumption of continuity also requires that the investment decision function has held for all time and that none of the parameters of the model have ever changed.

Conclusion

Presumably, the purpose of Kalecki's model is to open a window on the economy for a period during which time the model tells us something about the economy's business cycle. Yet Kalecki's solution of his fundamental equation leads directly to three basic problems: One, crucial economic elements of the model, c_1 and K(0), are lost in Kalecki's derivation and those missing elements influence the cycle implied

by the model; two, a slight change in the value of any single parameter renders the model unstable, and, three, the model's business cycle has no roots in the history of the economy. Thus, the cycles generated by the model are, in a real sense, arbitrary and disembodied. One can never be certain that they pertain to an actual economy.

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Notes

1. The full paper, entitled "A Mathematical Re-evaluation of Kalecki's Early Macrodynamics" (37 pages), is available from the authors or editor. The paper is currently under editorial review.
2. By 1943, Kalecki became convinced that cycle theory and growth theory were inseparable and that the twin economic phenomena had to be treated together within a single model. The interested reader may wish to consult Gomulka, Ostaszewski, and Davies (1990) on the economic limitations of Kalecki's later cyclical growth models. Their critique does not touch upon our more fundamental mathematically based criticism which is also applicable to Kalecki's later work.
3. Indeed, our approach to isolating the consequences of Kalecki's derivation has been for Baxley to write computer programs which permit us to implement Kalecki's analytic solution method and compare its results to those of our more general and more exact numerical method. Such an approach, of course, would not have been possible before the advent of modern computers.
4. Our solution is more complete because no terms are lost in the derivation. Retaining c1 and K(0) changes the nature of the cycles generated by Kalecki's model, using his parameter values.

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