A Study of J. M. Keynes' Marshallian-Pigouvian Elasticity Approach in Chapter 20 and 21 of the GT†

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In chapter 20 (and 21) of the GT, Keynes presented a complete microeconomic foundation for his macroeconomic analysis. Unfortunately, Keynes' analysis was overlooked. This led to the mistaken belief that Keynes' supply side analysis was confused and/or erroneous. Without such microfoundations, Keynes was judged as having failed to incorporate an analysis of rising prices/inflation within his model. This article demonstrates that this is not the case.

1. Introduction

Since the publication of his *Treatise on Money* (TM) in 1930, Keynes has been regarded as an inept microeconomist. In the *General Theory* (GT), his microfoundations of the aggregate supply function have been dismissed as confused, confusing or even erroneous. This paper reexamines chapters 20 and 21 of the GT and shows that such an assessment has no foundation in fact.

In 1930, Keynes realized that he had not made the microeconomic foundations of his aggregated Fundamental Equations clear enough. With the publication of Pigou's *The Theory of Unemployment* (TU) in 1933, Keynes realized *how* he could reformulate the TM's macroeconomic argument and combine it with a more rigorous microfoundation in explicit mathematical terms that would highlight his unique Aggregate Demand side analysis.

Pigou's approach in TU was in improved version of the Marshallian elasticity approach. Keynes used Pigou's approach and acknowledged doing so explicitly in his appendix to chapter 19 of the GT. Keynes then improved on Pigou's style by using a more consistent notation. For instance, where Pigou used X in his one sector model to stand for aggregate employment and X in his two sector model to stand for wage goods employment, Keynes used $N = \sum_r N_r$ for his aggregated model, where r = 1, 2. In his two sector model N_1 = consumption goods (wage goods) employment and N_2 = investment goods employment. Of course, this created a notational problem that Keynes resolved by redefining N to stand for wage goods employment in a special translation of his model into Pigou's notation, to be used in the appendix to chapter 19 only. If Pigou had used Z = X+y = aggregate employment, this problem would not have occurred. See Brady (1994b).

Keynes' mastery of the Marshallian mathematical approach to economics, elasticity analysis, is not well-known. This is due to Keynes' very concise and precise analytic style. In the GT, he used summation notation and subscripts to greatly reduce the amount of space he allocated to his mathematical economics exposition. From hindsight, this may have been a mistake.

At the time Keynes was writing, the knowledge of differential calculus and optimization theory, as applied to economics, was limited. This limitation can be seen when reading through the Harrod, Hawtrey[1] and Harris reviews of A.C.Pigou's TU. For instance, Harrod states that Pigou, "...contents himself with one reference in the Preface to those" writers on economic subjects, unacquainted with this tool [the differential calculus], who resent its use by others". (Harrod, 1934, p. 19). Hawtrey states that "In this analytical study of the theory of unemployment, Professor Pigou is breaking new ground. The treatment of the subject is predominately mathematical...Economists will be glad to have the opportunity of becoming acquainted with his methods and results at this stage." (Hawtrey, 1934, p. 147). Harris states "Chapters VIII to X (Part II) are...at once the most original, most difficult and most important analysis in the entire volume..." (Harris, 1935, p. 293) and "For the analysis, tho restricted to a relatively small number of variables and tho put in mathematical form, is most involved and difficult, and the introduction of a few more variable might well alienate the relatively few economists who have plodded through the book thus far." (Harris, 1935, p. 323).

The conclusion I draw from these reviews is that even the relatively few economists who had such training decided to skim chapters 8-10 in Part II of Pigou's TU. Hawtrey and Harrod barely scratch the surface of these very important chapters. Harris appears to be the only economist, outside of Keynes, who mastered Pigou's analysis.

It is interesting to note Harris' call for simplification. It should be noted that there is no graphical or diagrammatical analyses in Pigou's TU, except for *two* graphs in footnotes. In presenting the most advanced mathematics treatise on macroeconomic theory in the history of economic theory up through 1933, Pigou had challenged Keynes in a way that Keynes could not ignore. Of course, he responded with his GT. It is also heavily mathematical, with only *one* diagram. Keynes had decided to meet Pigou on his chosen battlefield of differential calculus, optimization theory and Marshallian elasticities. The author of *A Treatise on Probability* did not disappoint. Pigou was overwhelmed. Unfortunately, this "battle of the Titans" was only the beginning of the distorted portrayal of Keynes as being mathematically incompetent[2]. This strange conclusion has been allowed to become the "conventional wisdom" for 60 years. This paper will hopefully lead to a reappraisal of Keynes' contribution.

Following sections cover Keynes' derivation in chapter 20 of the crucial elasticity e_o, his unique way of presenting his supply-side results in terms of elasticities, the unique connection between the aggregate supply and employment functions of chapters 3, 6 and 20, and some possible reasons why Keynes' elasticity analysis, which incorporated the microeconomic foundations of his aggregated analysis, was overlooked.

2. The Roles of Elasticity of Demand, Price, and Marginal Revenue in the "What did Keynes mean?" Debate.

The reader might wish to reconsider some basic microeconomics. Given Total Revenue = TR = pq, where p is price and q is output, the MR function can be related to the elasticity of demand in the following manner.

Let $MR = dTR/dq = (pdq)/dq + (qdp)/dq$ (1)

$= p + q(dp/dq)$ (2)

$= p[1 + (q/p)(dp/dq)]$. (3)

Given $q = f(p)$, $e = (dq/dp)(p/q)$. Then

$MR = p[1 + (1/e)]$ (4)

Since e is negative, we can write

$MR = p[1 + (1/-e)]$ or (5)

$$MR = p[1-(1/e)] \tag{6}$$

In pure or perfect competition as $e \to \infty$, $MR \to p$. Confusion over whether Keynes understood this elasticity is a part of the "What did Keynes mean?" debate. Tarshis summarizes this part of the debate.

> Further on this subject: I have located lecture notes I made whilst attending lectures on 'The Short Period' given by R.F. Kahn in the autumn of 1932. It is perfectly clear that even then, in November 1932, Richard Kahn with Joan Robinson, had gone as far as was needed to provide the basis for Keynes' aggregate supply function in the sense in which I have used the term. Let me quote from my notes of Kahn's lectures to substantiate this view: 'In a world where businessmen behave rationally...' [but do they? There may have to be a minor qualification.] (Summarized by L.T. August 1976). 'If A = average value and M = marginal value of quantity and e = elasticity of demand, then A = M xe/(e-1). In equilibrium, MR=MC, and therefore price=marginal *cost* (in place of marginal revenue) multiplied by e/(e-1). And when e is infinite, P=MC' (lecture notes from 23 November, 1932). 'Price (to get maximum returns)=MCxe/(e-1)' (ibid., 30 November, 1932).
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> I suppose that one can argue till Doomsday about the significance of all of this. I see no way of doubting that Kahn and Joan Robinson had by then taken the subject as far as was needed; but how can we know that Keynes went along with them, or understood them? Perhaps it is correct, to quote from a letter that I have received on this point from Don Patinkin in which he says; '...it seems to me that your major argument is that my interpretation of Keynes could not be the right one for it makes Keynes appear "self-contradicting and wrong". And I say, that is precisely the case: on this point he was "self-contradicting and wrong". And, I say confused: witness his exposition in the 1934 draft....Accordingly, I feel that if you contend that in the final published form of the *General Theory* Keynes had escaped from this confusion - and that, in particular, he interpreted the ASF as you are now doing - then it is incumbent upon you to cite chapter and verse from Keynes' writings...to support your contention. Insofar as your own interpretation and concern, I can only repeat that the precise and (for that time) complex MR=MC analysis that you present here seems to me to be quite foreign to Keynes' general analytical style.'
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> Of course, I cannot know how much communication there was between Kahn and Keynes between say, 1931 and 1935, but my impression is that there was a great deal and I would be very surprised if Kahn would have allowed Keynes to submit to the publisher a concept of the aggregate supply function that was notably deficient and primitive as compared to what Kahn (and some others in the 'Circus') had developed as early as November 1932! (Tarshis, 1978, p. 62)

Equation (6) is easily reformulated into the form that Tarshis had in his lecture notes.

Let $1=(e/e)$. Then

$$MR = p[(e/e)-(1/e)] \tag{7}$$

$$= p[(e-1)/e] \tag{8}$$

Since $TR = p \cdot q$ and $AR = TR/q$, then $AR = (p \times q)/q = p$. So

$$MR = AR[(e-1)/e] \tag{9}$$

or $$MR/AR = [(e-1)/e]$$

We can now answer Don Patinkin's query. The "chapter" in which this analysis appears in the GT is chapter 20. The "verse" is on pp. 283-284. The specific elasticity that is identical to Tarshis's e is designated e_o by Keynes. We now present Keynes' analysis.

3. Keynes' "New" Formulation - The Employment Function of Chapter 20 of the GT

Keynes' understanding of elasticity analysis is demonstrated in chapter 20. We will concentrate on the elasticity e_w, which relates output to expected total revenue, proceeds or income.

For Keynes, TR=pq=D=p0. Starting from p. 280 and continuing through the middle of p. 285, Keynes works in wage units. Thus, $TR_w=p_wq=D_w=p_w0$. D_w is called expected effective demand (expected total revenue) because p_w is an expected price. Given Keynes' qualifications in footnote 3, p. 24 and footnote 1, p. 48 of the GT, Keynes' statement that expectations will be dealt with by the use of certainty equivalents allows Keynes to avoid any complications that would result from the introduction of probabilities into the analysis.

The reader should note that Keynes makes an abrupt change in his analysis, starting on the last half of p. 285 and continuing through the top of p. 286. Keynes starts working in *money*, not wage, units. This has caused some confusion for readers of the GT.

In the second line of footnote 1 on p. 283, after aggregating and using "d" instead of "Δ", we have Keynes' starting point:

"$dD_w = d(p_w 0) = p_w d0 + 0dp_w$". (GT, p.283)

Now divide by d0. We get

$MR_w = dD_w/d0 = p_w(d0/d0) + 0(dp_w/d0) = p_w + 0(dp_w/d0)$

$= p_w\left[1+(0/p_w)(dp_w/d0)\right] = (D_w/0)\left[1+(0/p_w)(dp_w/d0)\right]$

Then $(dD_w/d0)(0/D_w) = \left[1+(0/p_w)(dp_w/d0)\right]$ and

$(dD_w/d0)(0/D_w) = 1/e_0$ measured in wage units.

Thus, $1/e_0 = 1+(1/e) = 1-(1/e) = \left[(e/e)-(1/e)\right] = \left[(e-1)/e\right]$

Thus, $1/e_0 = MR_w/AR_w = \left[(e-1)/e\right]$, or in money terms, $1/e_0 = MR/AR = \left[(e-1)/e\right]$ or $e_0 = \left[(e)/(e-1)\right]$.[3]

4. The Inverse Function Rule and the Aggregate Supply Function Analysis in Chapters 3 and 6 of the GT.

Keynes presented the mathematical analysis of his Employment Function only in chapter 20, while in chapters 3 and 6 he provides only a bare-bones sketch of his Aggregate Supply Function. There are two reasons why he did this. First, in his TU, Pigou's formulations are all in the employment function form. Second, Keynes' emphasis on the *unique* relationship between Y_w and N on p. 90 of the GT, and the *unique* relationship between D_w and N (or Z_w and N since, at a maximum, $D_w=Z_w$,[4] so $p_w0=D_w=Z_w=P_w+N$) on pp. 280-282 of the GT means that Keynes is applying the Inverse Function Rule. Since the Employment Function is the inverse of the Aggregate Supply Function, the mathematical analysis need only be done once.

The inverse function rule states, in non-technical language, that for a function y=f(x) that meets certain conditions, a unique inverse function, x=f(y), exits. There are two cases. In case 1, if dy/dx>0, then dx/dy=1/(dy/dx)>0. The function $x = f^{-1}(y)$ is thus monotonically increasing. Thus, every value of x has a *unique* value of y *and* every value of y has a unique value of x. In case 2, dy/dx<0, and you have a monotonically decreasing function. Thus, for every value of Y_w, there is a unique value of N and for every value of N there is a unique value of Y_w. Similarly, for every value of $D_w(Z_w)$, there is a unique value of N and for every value of N there is a unique value of $D_w(Z_w)$.

In order to get the complete aggregate supply function analysis from the GT, it is necessary *for the reader* to apply the inverse function rule to the results of chapters 20 and 21.

Tarshis's (or Patinkin's or Samuelson's) reliance on chapters 3 and 6 alone is not sufficient for an understanding of Keynes' supply-side analysis. If we apply the inverse function rule, then all of the elasticities in chapters 20 and 21 have unique inverses in chapters 3 and 6. I will label these earlier elasticities with a -1 superscript. Thus, in chapter 20, Keynes' entire system, working in money units, is e_c, e_o, e_p, e_w, e_d and e, where e_c = (dN/dD)(D/N), e_o = (dO/dD)(D/O), e_p= (dp/dD)(D/p), e_w = (dw/dD)(D/w), e_d = (dD/dM)(M/D) and e = (dp/dM)(M/p), where N = total employment, D = total expected Revenue, O = total output, p = price, w = money wage and M = quantity of money.

These six elasticities can be reworked, in wage units, as Keynes demonstrated in the case of e_p and e_p'. The corresponding elasticities for chapters 3 and 6, in money units, are e_c^{-1}, e_o^{-1}, e_p^{-1}, e_w^{-1}, e_d^{-1} and e^{-1}, where e_c^{-1} = (dD/dN)(N/D), e_o^{-1} = (dD/dO)(O/D), e_p^{-1} = (dD/dp)(p/D), e_w^{-1} = dD/dw)(w/D), e_d^{-1} = (dM/dD)(D/M) and e^{-1} = (dM/dp)(p/M).

Since $1/e_o$ = the Kahn (Kahn-Robinson) elasticity, then $(1/e_o) = e_o^{-1}$. The aggregate supply function form of the elasticity that one obtains after applying the inverse function rule is nearly identical to the Kahn-Robinson elasticity cited by Tarshis from his 1932 lecture notes. The only difference is that Keynes' elasticities are first presented in wage units. At the middle of p. 285, however, all the elasticities are redefined in money units. For example, e_o = $(d0/dD_w)/(0/D_w)$ becomes e_o=(d0/dD)/(0/D). Taking the inverse of e_o, we obtain e^{-1}_o=(dD/d0)/(D/0)=(MR)/(AR). *Now* this *is identical* to the Kahn-Robinson formulation. Of course, Keynes' model involves 5 additional elasticities, 6 if we count the ratio e_c/e_o. This can be compared to Pigou's TU model, which involves 3 elasticities and the Kahn-Robinson model, which has 1 elasticity.

In 1934, Kahn *must have* realized that Keynes was reworking his 1933 drafts in terms of elasticities. Kahn incorrectly concluded that the final version of the GT would be based on *his* (Kahn's) elasticity analysis. However, Keynes was *not* patterning his analysis on Kahn's, but on Pigou's TU elasticity analysis. Since neither Kahn (nor J. Robinson) read or understood Pigou's elasticity analysis, presented in chapters 9 and 10 of Part II of TU, Kahn and Robinson jumped to the strange conclusion that Keynes was technically incompetent and mathematically illiterate. They then passed this incorrect evaluation down over a 40-50 year period to a host of other Cambridge economists[5]. I am not going to rehash the very numerous negative comments made about Keynes' technical skills by literally hundreds of economists over the last 60 years. However, see my footnotes 1 and 2. One conclusion is in order. Kahn's mathematical capabilities were not equal to those of Keynes[6] or Pigou. Thus, recent literature claiming that there is a "Keynes-Kahn" Theory of Effective Demand (see Chase, 1992, 1994)[7] is simply wrong. There is only a *Keynes* Theory of Effective Demand.

5. Why Was Chapter 20 Overlooked?

How could Keynes' Marshallian micro analysis, which, after a number of algebraic rearrangements and substitutions, appears straightforward, have been missed by the economists of the 1930's? Ten possible reasons suggest themselves.

The first reason is that Keynes defines his variables verbally. Thus, on p. 5 of the GT, Keynes does not write $w/p = MP_N = \phi'(N)$ or $w = vmp = pMP_N$, but says, "The [real] wage is equal to the marginal product of labor" or "the wage of an employed person is equal to the value which would be lost if employment were to be reduced by one unit". On pp. 17-18, Keynes does not write $0 = \phi(N)$, $\phi'(N)>0$, $\phi''(N)<0$, but says,

> "This is simply the obverse of the familiar proposition that industry is normally working subject to decreasing returns in the short period during which equipment etc. is assumed to be constant; so that the marginal product in the wage-good industries...necessarily diminishes as employment is increased".

Keynes presents most of his microdefinitions verbally. Thus, someone who is looking for a clear cut $w/p = \phi'(N) = MP_N$ will not find the result in the form he is looking for.

A second reason is that Keynes does not link his mathematical results by footnotes. Thus, pp. 44-45, pp. 55-56, footnote 2, p. 116, footnote 1, p. 283, footnotes 1 and 2, p. 285, footnote 1, pp. 280-286 and pp. 304-306 are all interrelated in terms of the mathematical analysis but appear isolated. An example of this is Keynes' critique of Pigou's mathematical analysis in the appendix to chapter 19. Keynes never comes out explicitly and says "compare my mathematical analysis in chapters 10 and 20-21 with Pigou's". Instead, he says that Pigou's analysis is very similar to some of his own concoctions, but is incomplete with respect to Keynes' own analysis, which incorporates the expenditure side.

A third reason, that appears to have created some confusion, is Keynes' use of a micro production function of the form $O_r = \psi_r(N_r)$ in chapters 4 and 6 and his use of a micro production function of the form $O_r = \phi_r(N_r)$ in the rest of the book, specifically in chapters 20-21. Keynes makes use of, or mentions, an aggregate production function explicitly only twice in the GT. The first place is on p. 209, which, however, is clearly related by Keynes to his discussion in chapter 21. But the chapter 21 discussion (pp. 304-306) is based on Keynes' chapter 20 analysis. Specifically, only on p. 285 of the GT does Keynes explicitly define an aggregate production function and it is a generalization of his two-sector microproduction functions, defined on p. 283 as $O_r = \phi(N_r)$; since "the same line of argument applies...where the elasticities without a suffix r apply to industry as a whole". Keynes expects the reader to realize that now $O = \phi(N)$. Once this is done, Keynes can derive $1-e_w(1-e_w) = e_p$, which holds at the macrolevel. Its microlevel counterpart would be $1-e_{wr}(1-e_{wr}) = e_{pr}$, where r=1=consumption goods and r=2=investment goods.

On p. 44, Keynes defines a two-sector micro production function $O_r = \psi_r(N_r)$. But on p. 45 he explicitly refuses to aggregate to the macrolevel, until further simplications are made. On pp. 44-45, all of the discussion is at the level of the firm. On pp. 55-56, ft. 2, however, Keynes is aggregating over all firms. Thus, there is no subscript, or, as Keynes calls it, suffix r. Keynes thought that his footnote 1 on p. 25 of the GT, linking chapter 3 to chapter 20, was sufficient notice to the reader that he should cover chapter 20 for a complete, formal analysis of the Aggregate Supply Function in its inverse form, the Employment Function.

We can now make the following conclusion. Whenever the subscript (or suffix) r appears in Keynes' mathematical notation, he is working at the microeconomic (firm or industry) level. Whenever there is *no* suffix r attached to a variable, he is working at the macroeconomic level. Unfortunately, Keynes waits until p. 285 to make this explicit although there is a hint on p. 45. If either chapter 20 or p. 285 is overlooked by a reader, the rest of Keynes' analysis will appear to be muddled.

A fourth reason is that, in order to work at the macro level, Keynes decided to use a framework which presents the result $w/p = \phi'(N) = MP_N$ in the form $p_w = p/w = 1/\phi'(N) = 1/MP_N$. In other words, the optimality condition is presented in an *inverse* form. Keynes assumed that the reader would realize that $p_w = p/w$ = the inverse of w/p. If an economist is looking for w/p, he will probably miss Keynes' version[8].

A fifth reason is that Keynes simply assumes that the reader is completely familiar with the Marshallian approach to microeconomics, so that Keynes is free to present his own Pigouvian concoction. Thus, Keynes simply skips the initial steps involved in setting up his optimization problem - defining the variables, objective function and constraints, maximizing his constrained or unconstrained objective function, deriving and economically interpreting the necessary first order and sufficient second order conditions for a profit maximum, and presenting a diagram of a strictly concave total revenue function, with a tangent intersecting it at one point from above, identifying the aggregate supply function with a slope of 1. This

view of Keynes is quite apparent from his reply to D.H. Robertson in Vol. 13 of the CWJMK, pp. 512-515. Keynes appears to be flabbergasted by Robertson's comments and states that what is involved is the standard micro theory of the supply function, with some new concoctions added. In Keynes' mind, he had already done all of the required initial steps in chapter two of the GT.

So, what was Keynes' new concoction? He simply takes the necessary first order condition for a profit maximum, $w/p = \psi'(N)$ and inverts it as $p_w = 1/\psi'_r(N_r)$ on p. 44 and pp. 55-56 (or as $1/\phi'_r(N_r)$ on pp. 283-285). He then multiplies both sides by the micro production function $0_r = \phi_r(N_r)$ (p. 283) and comes up with $D_w = p_w 0_r = \phi_r(N_r)/\phi'_r(N_r)$. This is identical to Pigou's approach in TU. To get to the macrolevel, he verbally writes that if one can assume a unit in which output as a whole can be measured, then one can aggregate over all industries, using an aggregate production function $0 = \phi(N)$ without the suffix r. Then you obtain $D_w = p_w 0 = \phi(N)/\phi'(N)$ on p. 285 of the GT in footnote 1. Again, Keynes is following Pigou. For Pigou, $I_w = p_w 0 = F(X)/F'(X)$. Under constant returns to labor, economic profit (K_w) equals zero so Pigou's $X = \phi(X) = p_w 0 = F(X)/F'(X) = I_w$, where $I = p0$.

A sixth reason is that Keynes forgot to explicitly define the elasticity for the aggregate production function after he gave his verbal definition of an aggregate production function. Thus, there is no explicitly defined result $(d0/dN)(N/0) = e'_0$. However, on p. 306 of the GT, Keynes presents a standard result, generalized from microeconomic theory, which is impossible to obtain unless just such an elasticity has already been defined.

A seventh reason is that Keynes leaves the impression, which is other than that which he intended to leave, from his comments on pp. 275, 297-298 and 305 of the GT, that mathematical analysis is irrelevant in economics. Keynes was opposed to the use of *purely formal* analysis, that is, the use of mathematical analysis which was divorced from actual economic application. Thus, unless the reader sees that Keynes is attacking what he regards as pseudo-mathematics, he is left with the wrong impression. Tarshis's Discussion comment is an example of precisely the wrong type of impression a reader of the GT can come to.

An eighth reason is that Keynes leaves out most of the intermediate steps involved in his mathematical analysis and only presents the final results. Thus, he usually gives only the first step, after which the reader is presented with the final result. Thus, the mathematical footnotes in the GT appear, at first glance, to be insignificant, minor results. In reality, a reader who attempts to work them out on his own quickly finds himself in deep trouble unless he understands his basic differential calculus, optimization theory and Marshallian elasticity approach perfectly. Given the low level of mathematical knowledge characteristic of economists in the 1930's, and the fact that few intermediate steps were included in the analysis by Keynes, it is not surprising that the vast majority of economists came to look upon the GT as a maze of pretentious and vague terminology, definitions and analysis that suggested the possibility that Keynes did not really understand what he had written.

The ninth reason is that Pigou's TU (1933) was not read by the vast majority of economists. Those few who did skipped the critical chapters, 8-10, of Part II. Keynes' belief that Pigou's technique had been absorbed was incorrect.

The final reason is that Keynes viewed chapter 3 as a strictly introductory treatment that might be "unintelligible" to some readers. Thus, the reader was asked to withhold judgment until *after* he had read the entire book and tied the analysis in chapters 10, 20-21 to the exposition in chapter 3. Unfortunately, it appears that a large majority of readers of the GT never took Keynes' warning to heart. Instead of chapter 20 being regarded as the *core* of his supply side analysis, the "wicked" chapter 3, which Keynes meant to be no more than a starting point, became the alpha and omega of his supply side analysis.

6. Conclusion

Keynes' elasticity e_o is identical to the elasticity formulation given by Kahn in a November, 1932 lecture to Lorie Tarshis and others. For whatever reason, Tarshis and his fellow students were misled into ignoring chapter 20 of the GT. Without chapter 20, the GT is indeed a demand side without a supply side, a theory capable of dealing only with a severe recession or depression. With chapter 20, the GT is a general theory of aggregate demand (AD)[9] and aggregate supply (AS) that easily explains the cases of rising prices and true inflation. Chapter 20 contains the microeconomic foundations of Keynes' macroeconomic theory.

With respect to Keynes' mid-1930's contributions to mathematical economics, the system of elasticities Keynes developed, for both his two sector and one sector models in chapter 20 of the GT, provided an advanced treatment of microeconomic theory, in relation to a macroeconomic analysis, that was at least equal in rigor to that provided by any other economist during the 1930's.

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Notes

1. Hawtrey's review is in error. He does not differentiate between the short-run and long-run. Pigou's Part II analysis is strictly short-run. Thus, Hawtrey's claim that Pigou's mathematical analysis on pp. 43-44 is open to criticism because "the assumption (?) that the rate of wage is equal to the marginal net product is only true if the employer asks for no contribution whatever towards net profit and overhead expenses from the output of the marginal man" is false. Letting $y = f(x)$ be a production function, Pigou defines $\psi(y)$ to be the demand (revenue) function. The demand price is then $d\psi/dy = MR$ = marginal revenue. $f(y)$ is the supply function. The supply price is then $df/dy = MC$ = marginal cost. Further, $dy/dx = f'(x)$ = the marginal product of labor = MP_x, so that $dx/dy = 1/MP_x = w/MP_x = w/f'(x)$.

Pigou's derivation of the necessary first order condition for a profit maximum in a purely or perfectly free economy, at the level of the firm, is $w = (d\psi/dx) - (df/dx)$, or $w(dx/dy) = d\psi/dx)(dx/dy)-(df/dx)(dx/dy)$, which is correct, since $(d\psi/dx)(dx/dy) = MR$, $(df/dx)(dx/dy) = MC_M$ = MC of the materials, and $w(dx/dy)$ = the marginal cost of labor = MC_x. At a maximum, be it an economic profit, normal (break even) profit, or loss minimizing outcome, $MR=MC_M+MC_x$ must hold. Given Pigou's assumption of diminishing returns to labor, this is also a sufficient condition.

Thus, Hawtrey's confusions on pp. 148-150 of his review indicates a clear lack of understanding of the theory of purely or perfectly competitive firms in input-output markets. This confusion on Hawtrey's part can best be grasped by reading through the Keynes-Hawtrey correspondence in Vol. 13 of the CWJMK. Hawtrey appears to have no understanding of an argument if it was presented in a purely theoretical form. His objections would deal with precisely those frictions that economic theory was excluding from consideration.

2. Recently, some Einstein biographers have assessed him as having been limited in his mathematical capabilities, with his wife, Mileva, helping him "solve certain mathematical problems" (Highfield and Carter 1993, pp. 114-115, 177). Nevertheless, Highfield and Carter remark that, "Any suggestion that Einstein was secretly incapable of understanding his own work would be bizarre", (ibid., p. 115).

The author finds the following series of statements, spanning 50 years, about Keynes to be highly questionable:

> "Again, as in the *Treatise*, Keynes did not really understand what he had written, and chose the wrong thing to publicize as his innovation. The Keynesian supply curve of labor and definition of involuntary unemployment were no more important to the *General Theory* than the "fundamental equations" were to the *Treatise*", (Klein, 1947, p. 83).
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> "Indeed, until the appearance of the mathematical models of Meade, Lange, Hicks, and Harrod there is reason to believe that Keynes himself did not truly understand his own analysis" (Samuelson, 1946, p. 188).

"...part of the blame belongs to Keynes himself and his collaborators because what Keynes meant can at times only be inferred by assuming that he was rational and then attempting to work out what he must have meant in order to arrive at the conclusions he did. It must be said that one of Britain's greatest writers did not quite cover himself with glory in Chapters 3 and 6 of the *General Theory*" (Tarshis, 1977, p. 54).

"*Tarshis*: We have that in our lecture notes, too. He spent half a lecture once on it. He felt that the stuff of economics was not sharp or precise, and it was too easy to distort it and create for it the impression of an exactitude that it really lacked, and by subjecting it to mathematical manipulation also to wind up with a seriously distorted picture of the economy. That also comes out in the *General Theory* and even more clearly in Vol. XIII of the *Collected Writings*. He had little patience with the use - or should I say abuse? - of mathematics in economics." (Discussion, 1977, p. 73).

"After Keynes died the tutor recovered his nerve a bit, and began to read the *General Theory* carefully and he found that it was full of the most frightful howlers (I will explain about the howlers in a minute)... Now I will explain to you about the errors in the *General Theory*" (Robinson, 1973, p. 259). (She never did.)

"This discussion culminated in Weintraub's 1957 article which Clower, in personal correspondence (dated 1 November 1957), characterized as 'a beautifully clear statement of what Keynes' "should have meant" if we suppose that he was a rational being.

The aggregate supply function as stated by Keynes and explicitly developed by Weintraub (1957), Davidson (1982)..." (Davidson, 1987, p. 51).

Since Keynes presented a completely sound analysis of his aggregate supply function and its microfoundations in chapter 20, and this analysis is the same as that presented by, for example Paul Davidson (1987) or S. Weintraub (1957) one could hope that economists such as Clower, Tarshis, Davidson or Solow (Harcourt, 1994, pp. 19, 21) would now conclude that Keynes was rational. Perhaps they were unfamiliar with Keynes' notation. See footnote 8. However, this appears to be a vain hope. Davidson, in his latest book, continues to search for Keynes' aggregate supply function and macroeconomics in chapter 3 of the GT. Davidson will have to look in chapter 20 for Keynes' explicit analysis (Davidson, 1994, chapter 10).

Patinkin, who has argued for over 45 years that Keynes' supply side analysis was either missing or in error, has made similar comments to those above. His position is well known; hence, there is no need to quote him. Patinkin, like Davidson, will need to reassess his position in light of the "new" evidence contained in chapter 20.

3. The reader can also obtain this result by direct inspection. Thus, given $0=F(D_w)$, then $(D_w/0)[(d0)/(dD_w)] = (1/MR_w)/(1/AR_w) = e_o$, so that $(MR_w)/(AR_w) = 1/e_o$.

Now let us work directly from the text of the GT on pp. 282-83, ignoring footnotes 1 and 2 on p. 283. After aggregating, we obtain $e_o = (d0/dD_w)(D_w/0)$ and $e_e=(dN/dD_w)(D_w/N)$

Now divide e_e by e_o. We obtain $(d0/dD_w)(D_w/0)\,(dD_w/dN)(N/D_w)$, which is equal to $(d0/dN)(N/0) = e'_o$.

Thus $e_e/e_o = e'_o$, $e_o=e_e e'_o$ and $e_e = e_o/e'_o$.

Since $d0/dD_w = (0/D_w)(dN/dD_w)(D_w/N)(d0/dN)(N/0)$,

then $dD_w/d0 = MR_w = (D_w/0)(dD_w/dN)(N/dD_w)(dN/d0)(0/N)$.

But $(D_w/0) = p_w$, so $MR_w=p_w\,(dD_w/dN)(N/D_w)(dN/d0)(0/N) =p_w\,(dD_w/d0)(0/D_w)$

$=AR_w(1/e_o)$ or $(1/e_o) = MR_w/AR_w$

If $e_e = 1$, then $MR_w = AR_w$, or constant returns to labor. This is exactly what Keynes states at the top of p. 284 of the GT and also on p. 306.

From $O=F(Dw)$, one obtains e_o. From $N=F(Dw)$, one obtains e_e. Keynes has no explicit production function elasticity. Rightly or wrongly, I believe Keynes assumed that, since he clearly defined $O=\phi(N)$, the reader would see that the production function elasticity, $(N/O)(dO/dN) = e'_o$, which Keynes omits, is simply the *ratio* (e_e/e_o). Thus, one can substitute $e_e e_o'$ on p. 305 for e_e as it appears in the e_p formula on p. 285. Once this substitution is made, all the analysis of p. 306 is clear and correct.

Let us skip past the problem of properly defining e_e *in the context of* chapter 21 and go directly to p. 306, which is a continuation of p. 285. On p. 306, Keynes states: "...if there are constant returns throughout so that marginal return equals average return, $e_e e_o=1$." (Keynes, 1964, p. 306).

This result which is identical to the result in chapter 20, is *impossible* to obtain if e_o is defined as the elasticity of output with respect to effective demand in money units, as on p. 285, since, by definition $e_e = (D/N)(dN/dD)$ and $e_o = (D/0)(d0/dD)$. Then $[(D/N(dN/dD)] \cdot [(D/0)(d0/dD)] = 1$ or $(D/N)(dN/dD) = (0/D)(dD/d0)$.

However, by definition, *this statement is false*, because $(D/N)(dN/dD) \neq (0/D)(dD/d0)$.

Therefore, under constant returns to labor, $e_e e_o \neq 1$. Now redefine e_o on p. 305 of the GT as $e_o' = (N/0)(d0/dN)$, which is the *ratio* of (e_e/e_o). Then $e_e \cdot e_o' = 1$ equals $[(D/N)(dN/dD)] \cdot [(N/0)(d0/dN)] = 1$ or, after canceling terms, $(D/0)(d0/dD) = 1$.

We get the following result, after we multiply both sides by (dD/d0): (D/0) = (dD/d0).
From pages 283, ft. 1 and 2 and 285, ft. 1, $D_w = p_w0$ becomes D = p0, given $p = p_w w$ and $D = D_w w$, where D represents Aggregate expected total revenue in money terms. Then, substituting for D in (D/0), we obtain [(p0)/0] = p after canceling 0.
Taking (dD/d0) of D, where D = p0, we also get dD/d0 = p. Thus, the average return equals the marginal return, just as Keynes stated, under constant returns. Keynes' "slip" is in not having explicitly defined the ratio $(e_w/e_w) = e_c$'. However, a study of p. 306 of the GT reveals all of Keynes' derived results to be correct, once e_w is replaced by e_w'. One result is incorrect. On line 9 of p. 306, the last e_w should equal 1, not 0.

4. A referee asks "what does it mean to say that $D_w = Z_w$ at a maximum"?" and "how can Z_w (the aggregate supply price of output) possibly equal P_w (profits) +N (employment)". Given that Keynes will "let Z be the aggregate supply price of the output from employing N men" (GT, p.25) and that "for the aggregate supply function for a given firm...is given by $Z_r = \phi_r(Nr)$ (GT, p. 45), then $Z_x = p_x O_r = p_x \psi_r (N_r) = P_r + {}_wN_r$.
In chapter 20, Keynes defines D=pO or $D_w = p_w O$. How can $Z_w = p_w O$ and $D_w = p_w O$? The apparent problem, which bothered Robertson and Hawtrey in 1956 (Robertson, 1955, p. 485; Hawtrey, 1956, p. 482-484) evaporates once chapter 20 is studied and it is realized that Keynes is applying standard optimization theory, i.e., maximizing expected profit, P or P_w. Given $D_w = p_w O$, where we use $O = \psi(N)$ as our production function and not $O = \phi(N)$, as in chapter 20, and $Z_w = P_w + N$, then $D_w = P_w \psi(N)$. Ignoring the subscript r, then $dD_w/dN = p_w \psi'(N)$ and $dZ_w/dN = 1$. Setting $dD_w/dN = dz_w/dN$, we get $p_w \psi'(N) = 1$. This is another way of writing the necessary first order condition for a profit maximum, which is normally presented as $\psi'(N) = w/p = MP_n$. Thus $Z(Z_w)$ *always* has a slope of 1. It is the equation of the Tangent line where it intersects Keynes' expected Total Revenue Function, a concave function under the assumption of diminishing returns, from above. P_w is the intercept term. Under constant returns to labor $Z_w = \phi(N) = N$, since $P_w = 0$. In this case, the aggregate demand function, aggregate supply function and aggregate supply *curve*, a locus of points satisfying the necessary first order conditions, are coincident. The aggregate supply curve would be linear with a slope of 1. Under diminishing returns, since $P_w > 0$, you get a *family* of Zw curves satisfying the necessary *and sufficient* conditions for a profit maximum. The aggregate supply *curve* would now be a convex curve with a slope greater than 1. It is a locus of all possible profit maximizing outcomes, given differing future expected prices. Thus, at every such point, you have $D_w = Z_w = P_w O = P_w + N$ for possible expected prices. At such points, *and only at such points*, Keynes can have $Z_w(Z) = p_w O(pO)$ as well as $D_w(D) = p_w O(pO)$. Keynes called these points points of effective demand. Given $Y_w = C_w + I_w$, the expenditure function, only *one* of these possible points can occur, given the marginal propensities to consume and invest.

5. A referee asks for evidence that J. Robinson and R. Kahn did not understand Pigou's (or Keynes') elasticity analysis. Let us start with Joan Robinson. In a letter to Keynes (1973a, p. 645) she stated that, with regard to the appendix to chapter 19 of the GT, in which Keynes compared-contrasted his mathematical economics with Pigou's, she "never got far enough with Pigou to be able to offer any remarks".
Let me remind the reader that Pigou's I is Keynes' D, Pigou's F(X) is Keynes' $\phi(N)$, Pigou's X is Keynes' N, and Pigou's K is Keynes' P. Since J. Robinson never got far enough to grasp these similarities, I conclude that Robinson did not understand Pigou's elasticities, since those elasticities are formed from the above variables.
How about Keynes' very similar elasticity analysis? Robinson states: "The third point at which Kalecki tightened up the slack in the GT was in connection with the relation of prices to wage rates. Keynes relied upon a rather vague sort of Marshallian concept of competition." (Robinson, 1965, p.97). Twenty years later she states: "Keynes did not bother much about "micro-theory"...and his idea of 'user cost' did not take on." (Robinson, p. 89). Finally, she is famous for claiming that G. Shove told her that "Keynes never took the twenty minutes necessary to understand the theory of value". If Joan Robinson understood that the elasticity analysis of Keynes contained his macroeconomics, just as it did for Pigou, she would not have made the strange statements that I have just presented.
Now we turn to R. Kahn. On 30 August, 1978, Patinkin sent Kahn a draft of a paper which Patinkin argued "contained material which supported earlier contentions of mine that Keynes' presentation of the aggregate supply function in the GT was confused and even erroneous" (Patinkin, 1993, p. 659). Kahn responded to Patinkin in a letter of 11 October, 1978. He states "I claim that I brought the theory of value into the GT in the form of a concept of the supply curve of output as a whole and that this was a major contribution" (Patinkin, 1993, p. 659). Patinkin responded with a letter of 30 October, 1978:

> "I am afraid, though, that there is some disagreement between us as to the specific importance of the aggregate supply curve in the GT. In any event, I find the discussion of this curve in the GT confused and, at least at one point, incorrect...What I find particularly puzzling is why your obvious expertise in these questions did not lead to the elimination of these confusions in the final, published version of the GT" (Patinkin, 1993, p. 660).

Kahn responded on 27 November 1978. He stated that "the points which you raise in your letter are precisely issues on which I am working at the moment. I must confess that I am puzzled in just the same way as you are" (Patinkin, 1993, p. 660).

Patinkin comments that "strangely enough, however, none of these points was discussed by Kahn in the published version of his lectures...there is no mention in it about bringing the theory of value into the *General Theory*. Nor is there a mention of the aggregate supply function" (Patinkin, 1993, p. 660). Patinkin continues:

> "Finally, though in his sixth lecture on 'personal relations with Keynes', he does include a two-page section on 'my part in the *General Theory*'...Kahn then concludes that his written contributions 'were made in the margins of galley proofs which have not survived...and that his major contribution was made in the course of personal discussions with Keynes" (Patinkin, 1993, p. 661).

Patinkin then states that "I see no reason not to accept the evidence of his foregoing letters to me and to conclude that Kahn regarded his major contribution to the *General Theory* as being...the aggregate supply function...This conclusion accords nicely with the fact that the properties, of the supply curve...had been the major concern of Kahn's 1929 Fellowship Dissertation" (Patinkin, 1993, p. 661).

Unfortunately Patinkin was blinded, like Samuelson, by his friendship with Kahn. The "evidence of his foregoing letters to me" supports precisely the *opposite* conclusion, that Kahn did *not* contribute the aggregate supply curve (or function) analysis in the G.T.!! If Kahn had, he would have immediately pointed out to Patinkin the fact that Pigou's analysis of the supply side is identical to Keynes' one sector analysis and practically the same for the two sector analysis. But Kahn *never read Pigou*. Thus, he *could never* respond to Patinkin's letter of October 30, 1978. All the evidence in the Kahn-Patinkin exchange supports the conclusion that Keynes patterned his analysis of the supply side, not after Kahn's 1929 Fellowship Dissertation, but after Pigou's 1933 *TU* analysis.

Robinson and Kahn likewise were confused about the need for introducing imperfect competition into Keynes' Theory of Effective Demand. There is no need to, since involuntary unemployment has nothing to do with imperfect competition. The general equilibrium condition in the GT, $w/p = MP_n/(MPC + MPI)$, is easily generalized for the case of imperfect competition to $[1-(1/e_d)]\ w/p = MP_n/(MPC + MPI)\ [1 + (1/e_s)]$, where e_d = elasticity of demand and e_s = elasticity of supply.

In conclusion, the claim of Patinkin (1994, 1144-45) that Kahn and Robinson "played an important role in providing Keynes with critical comments on the successive drafts of the book" has no support. Even more off-base is Earley's claim that "The General Theory was written with the close collaboration of a group of colleagues" (Earley, 1994, p. 27), or Tarshis' claim that "part of the blame belongs to Keynes himself and his collaborators because what Keynes meant can at times only be inferred by assuming that he was rational and then attempting to work out what he must have meant in order to arrive at the conclusions he did" (Tarshis, 1977, p. 54).

Everything is worked out in chapter 20. Unfortunately, Patinkin, Tarshis, Samuelson, and others overlooked chapter 20. It is in this chapter "where the micro-foundations of (Keynes') macroeconomics are located" (Tarshis, ibid., p. 54).

6. A referee notes that the general view I take concerning the role that mathematics played in Keynes' GT is similar to that expressed by O'Donnell (1989, 1990). The reader can compare these sources with Brady (1983, 1988). In general, the referee is correct. However, there is one very specific difference concerning the assessment of the mathematical exposition in Chapters 20 and 21 of the GT which involves the Keynes-Townshend correspondence of 1936. Based on this correspondence, O'Donnell concludes that Keynes' view of "the subordinate role of mathematics was emphasized by his disinclination to render the algebra of chapters 20-21 of the GT logically water tight and his preference for eliminating it altogether in any future revision" (O'Donnell, 1989, p. 197; 1990, p. 42). Of course, the calculus and optimization theory that Keynes used in chapters 20 and 21 is watertight except for an error introduced into the price equation by Keynes on p. 44 of the GT in a second British printing. Instead of adding user cost, U_r, Keynes, if he were to add anything, should have added marginal user cost, ΔU_r, or MUC_r, where MUC equals marginal user cost. Keynes erred in taking Townshend's advise. Keynes also neglected, on p. 300 of the GT, to explicitly define the ratio e_s/e_d. Secondly, Keynes suggested eliminating his analysis only in the context of a future revision which presented his breakthroughs in a positive manner and which would not have required him to make the negative comparison-contrast between his theory and Pigou's. In 1936, Keynes had not yet grasped how dear the technical mathematical representation of an economic argument was becoming to economists in general. If Keynes could have foreseen Samuelson's 1946 "eulogy" of him, I believe Keynes would have put in all of the missing intermediate steps of his chapters 20-21 analysis into a second edition and more. Much more.

On the other hand, Skidelsky's view of Keynes' mathematical capabilities suffers from his reliance on what appears to be private conversations with Richard Kahn. It also lacks the sophistication of O'Donnell's overall view. Skidelsky states that Kahn "recalled Keynes himself as being a poor mathematician by 1927; his ability

to think mathematically was to be of crucial help to Keynes in the early 1930's...His relationship with Keynes was a prototype of the other important relationships of his life, particularly with Joan Robinson" (Skidelsky, 1992, p. 288; see also pp. 412-413, 423-426, 443, 538-541, 610-614, and especially his footnote 50 on p. 426). Of course, the *one* time, and the only time, that Keynes asked for Kahn's aid in a mathematical analysis, in late 1931, Kahn replied: "I must apologize...Even now I am afraid I have nothing useful to say. The difficulty is that it will take me a little time to get at home again with curly d's...I feel more that this new method is the right one...but I have not quite been able to follow the steps" (CWJMK, Vol. 13).

There is no support in the literature for any of Skidelsky's claims or musings. He has simply taken Kahn's (and Robinson's) musings at face value without a shred of critical scholarship. Skidelsky (1992) is merely continuing his incorrect assessment of Keynes' mathematical views, first presented in 1983 (Skidelsky, 1983, pp. 222-227). He still does not know the difference between the pure mathematics of G.H. Hardy and J.E. Littlewood, the applied mathematical economics of Marshall, Pigou, and Keynes (elasticity analysis) and the wholly applied mathematical approach of the Cambridge mathematical tripos taken by Keynes in 1905 (See Mordell, 1970; Bur Kill, 1979; Roth, 1971; Forsyth, 1935).

If Keynes was not considered to be a fine applied mathematician, he would not have been asked to review R. Fisher's 1935 paper in the *Journal of the Royal Statistical Society* (O'Donnell, 1989, p. 207). Keynes' qualification was his February 1911 paper, published in the above mentioned journal, concerning "what function of these measurements will yield us the *most probable* value of the quantity, on the basis of this evidence" (1921, p. 194). Keynes' paper was a very important contribution to the "maximum likelihood estimate" approach of Fisher (1922). Surprisingly, Savage, while acknowledging Edgeworth's less significant contribution as being a precursor of Fisher's approach (Savage, 1976, p. 459), is silent about Keynes' much more significant contribution.

In conclusion, Kahn's assessment that "Keynes was a poor mathematician by 1927" was not shared by the editor of the *Journal of the Royal Statistical Society*, who selected Keynes to review Fisher's paper. Kahn *never* gave *any* examples of Keynes' alleged mathematical deficiencies in his lifetime. If Keynes had been a poor mathematician by 1927, it would have been a technically impossible task for him to have written the *TM* or *GT*, or have read A.C. Pigou's *TU* (1933).

7. Chase repeats the Clower-Klein-Samuelson-Tarshis-Kahn-Robinson-Patinkin view about Keynes. In his 1992 article, we discover that Keynes' supply side analysis in the GT is "obscure, imperfect, and/or incomplete" (Chase, 1992, p. 876), that Keynes was a two-faced Janus whose GT must be carefully interpreted, "clarified and elucidated", (Chase, 1992, p. 877, 875). In the 1992 paper, Keynes is still regarded as being the author of the Theory of Effective Demand. However, Chase, in his 1994 article claims: "Because the linkage between Keynes' microeconomics in the TM... and his macroeconomics in the GT...was never self-consciously developed by him..." (Chase, 1994, p. 85). From this, he then starts talking about "...the Keynes/Kahn Theory of effective demand..." (Chase, 1994, p.852), "...the gap between his (and Kahn's) theory of effective demand (c. 1932-33)...", (Chase, 1994, p. 856), "...the adjustment mechanism of aggregate demand (the macroeconomics of the Keynes/Kahn theory of effective demand..." (Chase, 1994, p. 863). There is simply no evidence to support these claims. Brady (1994a) demonstrated that the microeconomic foundations of the *Treatise on Money* (TM) and GT are identical. Thus, the claim that Kahn was a co-author of Keynes' Theory of Effective Demand is simply not supported by the evidence, unless one is confusing the theory of the employment multiplier with Keynes' theory.

Mankiw repeats the Samuelson-Klein theme that the GT "is an obscure book. I am not sure that even Keynes himself knew completely what he really meant" (Mankiw, 1992, p. 561). Samuelson, in his latest assessment (Samuelson, 1994a, 1994, b) still holds to his strange 1946 claim that Keynes did not fully or completely understand what he had written. He still presents the theory of effective demand as having no microfoundations or aggregate supply curve (or function) (Samuelson, 1994b, p. 64). He does not distinguish between current realized income $C_w + I_w$ (Keynes' Y_w) and future expected income (Keynes' D_w). Thus, PQ = I + C only if Say's law holds, where PQ = D and I + C = Y, where M = D/V(r), not M = PQ/V(r), etc. The entire "mathematical version of Keynes" presented by Samuelson is incorrect, because it does not incorporate the decision makers expectations, in my opinion.

Samuelson states that "there is delicious paradox in the fact that every Tom, Dick, and Harry who read *The General Theory* could formulate the full system...after the fact, whereas the Master himself only intermittently thought in its explicit terms (and several times Keynes said or wrote things that revealed distinct Homerian nods!). Hardy and Littlewood were mathematical snobs, but they were right that 13th Wrangler Keynes lacked comparative advantage in mathematics" (Samuelson, 1994a, p. 70). This makes no sense.

Contrary to Samuelson, a comparison-contrast of Meade's 1937 *Review of Economic Studies* two sector model with the GT model of chapters 10, 20-21 reveals that they are identical with one very important difference. Keynes mathematically distinguishes expected future income, D_w, from realized current income, Y_w. Since

Hicks and Harrod based their IS-LM graphics on Meade's model, an economist can just as easily derive IS-LM from Keynes' model. Thus, we now have the answer as to why Keynes "had no criticism of Meade's presentation" (Skidelsky, 1992, p. 614). Meade's two-sector model is nearly the same as Keynes'!!! Hicks' reply, about only including realized income and ignoring expected income in his IS-LM model because "...I feel much happier if it is put in and marked unreliable, than if it is merely talked about, and not...put into the formula, which [the student] will take down in his notes" (Skidelsky, 1992, p. 615) reveals Hicks' ignorance of Keynes' chapter 10, 20-21 model. A copy of this comparison-contrast between Keynes and Meade is available from the author on request.

Samuelson then presents four pieces of "evidence" in support of his claim that "I believe Richard Kahn's originality to be at least the equal of Maynard Keynes'" or "Kahn alone, in his signed work or informal communication with Keynes - or Kahn *and* the circus (particularly Joan Robinson and James Meade) - arrived as early as Keynes did at an understanding of how underemployment income would rise, fall, and stay in equilibrium as a result of the interaction between propensities to consume or save and to invest", or "that simple income analysis of PQ determination is as much the innovative creation of R.F. Kahn as of J.M. Keynes" (Samuelson, 1994a, p. 20). Again, there is not a shred of relevant data to support Samuelson.

Let us examine this "evidence". The first three pieces are the unsupported claims of Schumpeter, Luigi Pasinetti, and Michael Poston's "common room gossip of the university". The fourth piece of "evidence" is, *naturally*, Samuelson's meeting with Joan Robinson and Richard Kahn in 1948, after receiving a letter "Joan wrote marveling that *a priorism* could hit so near the mark" (Samuelson, 1994, p. 65). Concerning Samuelson's statement that "he [Keynes] entered it [the Great Depression] the sort of man who might be expected to embrace *the General Theory* if it were explained to him. From the previous record...one cannot say more" (Samuelson, p. 65). In other words, based on Joan Robinson's story that Kahn was the one who explained the GT to Keynes. Eliminating this fantastic fantasy of Joan Robinson, Samuelson is left with no relevant evidence.

8. A referee argues that "again, only a dunce would be confused by Keynes' procedure". I agree. However, overlooking something and being confused are two separate issues. Thus, Friedman (1974, pp. 156-157) clearly overlooked Keynes' inversion. He states, in a footnote covering two pages, that "Keynes defines: p_w = price of goods in wage units...$p = p_w W$ = price of goods in money...!!

If Friedman (or Tobin, 1974, p. 79 or Davidson, 1974, pp. 93-94) had understood what Keynes, who was only following Pigou's innovative approach, was doing, he would have written P_w = price of goods in wage units = $p/W = 1/\phi'(N) = 1/MP_N$; and $p = p_w W$ = price of goods in money = marginal cost. Similarly, Friedman states that "From (1) and (2), $p = [D_w W]/0$".

Of course $[D_w W]/0$ = marginal cost since $D_w = p_w 0$ or $p=[p_w\ OW]/0 = p_w W$, which is identical to Keynes' exposition on p. 44 of the GT. I conclude that Friedman, Tobin, and Davidson were unfamiliar with Pigou's original exposition of his macroeconomics because they overlooked Part II of Pigou's book. All three then started to write on Keynes' GT minus any understanding of the role chapter 20 played in the GT. They then past this assessment down to their students.

9. A referee asks about the confusion concerning "whether sales proceeds were realized or expected in chapter 3 of the GT?" The answer is that D, in chapter 3, as well as in every other chapter of the GT, equaled expected proceeds, or revenue or income. $D_1 = p_1O_1$ was designated to be the expected proceeds from the expected sale of consumption goods "based on short run expectations about p_1, the expected price of currently produced consumption goods. $D_2 = p_2O_2$ was the expected proceeds from the sale of investment goods, O_2, based on long run expectations about p_2 the expected price of currently produced investment goods. $D = D_1 + D_2$ thus integrates both long and short run expectations into Keynes' analysis. However, in order to simplify the analysis, and to avoid having to resort to his logical theory of probability or his applied theory of evidential weight, based on his "conventional coefficient of risk and weight" analysis, Keynes, in footnote 3 on p. 24 (p. 48) resorts to a "certainty equivalent" assumption. Thus, D_1 and D_2 are certainly equivalents. They are *not* realized results.

On the other hand, Y is defined to be realized income, proceeds, or returns based on actual propensities to spend in chapters 8 and 10. As Keynes states explicitly in his discussion on pp. 77-79 of the GT, the difference between D and Y is what the GT is all about, as opposed to his TM, where the distinction was not made. If short run expectations are realized, then $C = D_1$. Keynes' argument is that it is long run expectations about long lived durable capital equipment that are seldomly realized, so that $I < D_2$. Then $Y < D$ and an unemployment gap that is involuntary arises. Assuming short run expectations are realized changes nothing. There is only *one* realized expenditure function, $Y = C+I$. On the other hand, there is a *family* of different expected profit maximizing D and Z functions, a different D and Z function corresponding to each different expected price or each different expected profit. The set of all expected D and Z functions is bounded from

above by a particular D and Z corresponding to the maximum "potential available" employment and its maximum potential production. We will designate these as D* and Z*. According to Keynes, they are obtained, if and only if, the elasticity $e_p = 0$. Then $D^* = Z^* = Y$ and there will be no gap between the potential and actual full employment level of output for an economy. This will only obtain, however, if $p = p^*$ and $P = P^*$, where p^* and P^* are the optimally expected price and economic profit corresponding to D* and Z*.

The reader should note that, except in the case of constant returns to labor, the aggregate supply curve (a.s.c.) is distinct from the aggregate supply function $Z(Z_w)$. $Z = P + wN$ ($Z_w = P_w + N$) will always be linear with a slope of 1 ($dZ_w/dN=1$) or a slope given by the reciprocal of the money wage, w, ($dZ/dN = w$ or $dN/dZ = 1/w$) depending on whether you are using the employment or aggregate supply function and depending on whether you are working in wage or money units.

The a.s.c. is a locus of all expected profit maximums. It is obtained by taking the derivative of $D_w = p_wO(=Z_w = p_w + N)$ which is $dD_w/dN = d/p_wO)/dN = d[\phi'(N)/\psi'(N)]/dN$ which has a slope greater than one in the diminishing returns case and a slope equal to one in the constant returns case. Under constant returns, Z and the a.s.c., D=Z, all have a slope of one and are coincident. Under diminishing returns, Z has a slope of 1 while the a.s.c. ($D_i = Z_i$, i = 1,..n) has a slope greater than one. Thus, Keynes' GT model in chapter 20 has *four* distinct functions. The first is Y, which equals actual realized expenditure or actual aggregate demand. The second is D, which equals expected expenditure or proceeds. The third is Z, or the expected necessary minimum proceeds which will satisfy the necessary first order condition for a profit maximum. The fourth function is the locus of all possible points of effective demand, $D_i = Z_i$, i = 1,..n. Keynes calls this fourth function his a.s.c.

Given the above, we can compare P. Davidson's handling of what the post Keynesians and English neo-Keynesians call AD-AS, or D-Z, analysis with Keynes' model of chapter 20. Davidson, whom I believe is a very representative example of the above mentioned schools, works *only* with two functions. He defines D to be the sum of the actual realized money consumption and investment expenditures. He has thus conflated the realized expenditure function, Y, with the expected proceeds function, D. Similarly, Davidson conflates $Z(Z_w)$ with the a.s.c., a locus of a number of D=Z intersections. Thus, he confuses the demand side, given by the expenditure function Y, with the supply side, given by D and the locus $D_i = Z_i$, i = 1,..n. Since Davidson, in effect, is using only two of Keynes' four functions, he can't possibly get the modeling of expectations down accurately.

E.R. Weintraub has stated that:

"The literature is full of arguments to the effect that Keynes had never really worked out, or did not understand, his own theory. Thus a students reading of Keynes was positively discouraged by many Keynesians on the grounds that more recent work had cleared up the many inconsistencies" (Weintraub, 1975, p. 541).

This failure to comprehend the technical developments undertaken by Keynes in chapter 20 of the GT have led to assessments like the following one:

"Thus, the attempt of Lord Keynes to remove inflation from his simplified model of a national economy in *The General Theory* led many of his followers to erroneous and very harmful conclusions about desirable governmental policies. The inability of the Keynesian model to handle the inflation problem adequately has brought increasing criticism of the model and has led to its displacement as a useful tool in solving current economic problems" (Quinto, 1982, p. 137).

Of course, such bizarre assessments hold only if chapters 20 and 21 of the GT are overlooked.

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