A Comparison-Contrast of J. M. Keynes' Mathematical Modeling Approach in the General Theory with some of his General Theory Interpreters, especially J.E. Meade

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The analysis in this paper demonstrates that J.M. Keynes' original mathematical model, presented in Chapters 10, 20-21 of the General Theory, is superior, in terms of mathematical completeness and rigor, to any other model put forward since 1936 which purports to give a technical exposition of "What Keynes Meant". This paper covers the mathematical models of Meade, Hicks, Harrod, Patinkin, and Barro-Grossman, as well as improved versions of the Meade model offered by Darity-Cottrell and Rappoport.

1 Introduction
How can there be so many differing interpretations of "what Keynes meant" or "what Keynes meant to say" in the General Theory (GT)? Meltzer gives the following answer:

"A third source of evidence available to economists comes from the inferences that the author claims to have drawn from his model. For mathematical models, these inferences are readily checked. Since Keynes did not set out an explicit set of equations, later readers have to do this work for themselves. They can check their formulation against the implications claimed by the author. I used this test for my models of both the Treatise and the General Theory." (Meltzer, 1992, p. 155).

Continuing this theme, Meltzer then states that

"First, the interpreters have different purposes. Some offer a particular interpretation, then show what Keynes would have had to say if he shared their interpretation. I cite Paul Davidson, Clower, and Leijonhufvud as three of many examples of writers in this tradition. Others develop a model or models of Keynes' theory, draw its implications, and compare the model's implications to the implications Keynes draws based on his verbal arguments. I am one of the latter." (Meltzer, 1992, p. 159).

Meltzer's argument would be correct if in fact "Keynes did not set out an explicit set of equations" and the implications Keynes drew in the GT were "based on his verbal arguments" alone. However, contrary to Meltzer and other economists who have written on the GT, Keynes did set out an explicit set of equations. Further, the implications he drew in the GT are based on and follow from the results of the mathematical analysis of his system of equations. Keynes, of course, also provided excellent verbal analysis as well in Chapters 2, 3, 6, and throughout the GT. Let me be quite clear that I am not claiming that those economists who have written on Keynes' GT do not understand Keynes' general point of view. Many
economists understand, in very general terms, what Keynes' major points were. What has been overlooked is the very precise and clear cut basic mathematical analysis of Keynes which supports these general points. Economists do not understand the logical, analytic, technical structure of the model of the GT.  

Why is this the case? What accounts for the failure of the economics profession to grasp the mathematical structure erected by Keynes 60 years ago? The answer I give is that between 1933 and 1939, the economics profession started out on a third revolution, besides those of Keynes and imperfect competition. This third revolution was the mathematization of the language used to express economic analysis by academic economists in journals and textbooks.

Keynes' GT, like A.C. Pigou's *The Theory of Unemployment* (TU), was too difficult for the vast majority of academic economists to understand during the transition period of this language revolution in economics. The one in a hundred economists who might possibly have understood these books never read them, since macroeconomics, or aggregated behavior, was not their field of specialization.

Thus, Skidelsky's statement that "It is significant that he refused to present the 'model' of the *General Theory* in mathematical form, even though he assembled its (verbal) elements in Chapter 18" (Skidelsky, 1992, p. 540) illustrates the horribly incorrect assessment of the GT that stretches back to Samuelson in 1946, fifty years ago. Economists have simply refused to go beyond the verbal analysis of Keynes' model presented in Chapters 2, 3, and 6 of the GT and consider the rigorous supply side presentation of Chapters 20 and 21, combined with the expenditure side in Chapter 10.

The comparison-contrast Keynes undertook in the Appendix to Chapter 19 of the GT with Pigou's TU would have resulted in a real revolution in pure economic theory, since Keynes successfully integrated his distinction between short-run expectations (SRE) and long-run expectations (LRE), not only into his aggregated analysis of the nominal commodity, labor and money markets, but also into the microtheory of the firm-industry. Keynes' theory of a dynamic shifting economy with many multiple equilibria, based on the feedback effects of failed producer expectations of the future on current behavior, would truly have revolutionized the way economists analyze economic behavior. It did not happen that way. Instead, Keynes was portrayed as a victim of schizophrenia by mathematically inept, illiterate, and innumerate economists. D. Robertson (Robertson, 1956, p. 485) and R. Hawtrey (Hawtrey, 1956, p. 484) concluded that "the distinction between expected and actual demand is, therefore, not material to Keynes' analysis". This is simply wrong. Of course, Robertson and Hawtrey were led to their incorrect assessments due to their inability to master the technical analysis of Chapters 20-21 of the GT.

The paper will develop the following framework. First, we will cover Keynes' mathematical model as developed in Chapter 20. We will derive the graphical analogue of Keynes' aggregate supply curve (a.s.c.) from the mathematical analysis of his a.s.c. in Chapter 20 of the GT. We will then show how his aggregate labor market model is derived from the a.s.c. Both diminishing returns and constant returns to labor will be analyzed. We will then turn to Meade's 1937 model of the GT and show how Keynes' model simplifies to Meade's model once we assume that all expected results are realized for both SRE and LRE. We will then look at those portions of the work of Patinkin and Barro-Grossman that bear on the issue of Keynes' mathematical modeling of his multiple unemployment equilibria model and the cause of involuntary unemployment. Finally, we will consider the Darity-Cottrell (1987) and Rappoport (1992) extensions of Meade's model. A concluding section follows.
2 Keynes' Mathematical Model of the GT - The Supply Side

Keynes presents both one and two sector models in the GT, the one sector model being an aggregated version of his two sector model, which contains a consumption goods sector and an investment goods sector. This is the same approach Keynes took in *The Treatise on Money* (TM) with his fundamental equations.

Keynes defines $p_r$, $r=1, 2$ to be the expected prices of consumption and investment goods. Thus,

$$p_1 = \text{the expected price of (short-run) consumption goods},$$

and

$$p_2 = \text{the expected price of (long-run) investment goods},$$

where $\Sigma p_r = p = p_1 + p_2$ is the aggregated price subject to Keynes' concerns about the problem of weighting prices correctly.

Using the standard microeconomic theory of perfectly, or purely, competitive firms, the necessary first order conditions for a maximum for a profit optimization problem are $p_1 = w(dN_1/d\sigma_1)$ and $p_2 = w(dN_2/d\sigma_2)$, where $w$ is a uniform, fixed short-run money wage in both sectors.

Keynes defines $N_r$, $r=1, 2$ to be the amount of employment in the consumption and investment goods industries. Thus,

$$N_1 = \text{The amount of employment in the consumption goods industries,}$$

and

$$N_2 = \text{The amount of employment in the investment goods industries.}$$

Thus, $N = \Sigma_i N_i$, $i=1, 2$ or $N = N_1 + N_2$ is aggregate employment, subject to Keynes' concern about correctly aggregating (see Chapter 4, pp. 41-44, GT). Keynes defines $O_i$ to be the amount of consumption goods output, where $O_i = \sigma_i(N_i)$ is a production function and $O_2$ is the amount of investment goods output, where $O_2 = \sigma_2(N_2)$, is a production function. If there is a unit of measurement available to reliably add $O_1$ and $O_2$, then

$$0 = \Sigma_i O_i, \ r=1, 2 \text{ or } 0 = O_1 + O_2,$$

where $0 = \sigma(N)$ is an aggregate production function using Chapter 20 notation. The reader should note that in the notation of Chapters 3, 4, and 6, $0 = \psi(N)$.

Due to Keynes' concerns about aggregating individual prices and outputs in a reliable and accurate manner, he decided to work in nominal terms. Keynes defines expected total proceeds, revenue, or income as

$$\Sigma_i D_i = \Sigma_i p_i O_i, \ r=1, 2.$$ (4)

Then $D_1 = D_1(SRE)$ and $D_2 = D_2(LRE)$, $D_2$ is the more important in determining aggregate employment.

Keynes defines $\Sigma_i Z_i = \Sigma_i P_i + wN_i = Z_1 + Z_2$,

where $Z_1 = P_1 + wN_1$

and

$$Z_2 = P_2 + wN_2.$$ (5)

$Z_r, r=1, 2$ is the expected return that is necessary, or must be attained, in order to maximize $D$. It represents the necessary first order condition for a profit maximum. $P_r$ is the expected economic profit. For each different expected price, there is a different level of expected effective demand, expected profit and a different level of employment. Full employment occurs where $e_r = 0$ or equivalently, where the a.s.c., a locus of all possible different $D_i = Z_r, i=1...n$, tangencies, becomes completely inelastic. Keynes calls this level of
employment and nominal output "maximal", "optimal", "maximum", or "optimum" at different points throughout the GT.

3 Keynes' Mathematical Model: The Demand Side

Keynes defines $C = \text{the actual or realized income from the sale of } O_1 \text{ consumption goods resulting from the utility-maximizing (or disutility minimizing) expenditure decisions of consumers.}$ Keynes defines $I = \text{the actual returns from the sale of } O_2 \text{ investment goods resulting from the expenditure decisions of producers.}$ Keynes defines $Y = C + I = \text{the total actual expenditures made by consumers and producers.}$ Note that if $C = D_1$, SRE are satisfied while if $I = D_2$, LRE are satisfied. Thus, in the aggregate, for all expectations to be realized, $C + I = Y = D_1 + D_2 = D = Z_1 + Z_2 = Z$ at $e_o = 0$. Only if this condition holds will the rationality postulate of optimizing firms and consumers, aggregated from the microlevel, be obtained and Say's Law would hold.

Keynes' argument is that the LRE of producers, with respect to $D_2$, are seldom, if ever, realized and $I > D_2$. This is due to the constant problem of technological obsolescence brought on by technological change. It is essentially "caste in concrete", is irreversible, and represents an irrevocable decision on the part of producers, based on LRE which are highly uncertain. These decisions can't be checked against realized results, as is the case for consumption goods, with downward/upward revisions in price, ceteris paribus, eliminating shortages/surpluses. Instead, the initial decision to purchase investment goods is subjected to a very high rate of discount. Thus, the return, which involves a series of expected profits over a number of successive future time periods, turns out to be less than $D_2$. The adjustment process leads to a deficiency in the purchase of new capital equipment, a decrease in the size of the optimal plant, a decrease in employment, a GNP gap and hence involuntary unemployment. This deficiency in the provision of long run durable capital formation, in both the private and public sector, is the problem of a capitalist system. It can't be dealt with by "fiscal and monetary policy". The gap between $C$ and $D$ is generally self-correcting, given more or less time, since the SRE, and needed adjustments in stocks, are continually correctable in the light of the most recent realized results. This is not the case for LRE and the gap between $I$ and $D_2$. This is the crucial problem.

Keynes defines $C_w = \chi(Y_w)$ or $C = w\chi(Y_w)$, where $\Delta Y_w = AC_w + \Delta I_w$, so that $\Delta Y_w = k\Delta I_w$ and $dC_o/dY_w = 1 - 1/k$ equals the marginal propensity to consume (m.p.c.) and $k$ = the investment multiplier.

Keynes unfortunately waits until page 209 to formally define $Y = P_0$, where $P$ equals the actual, realized price of aggregate output $O$ that is sold. I am going to rewrite this as $Y = P_0$ since Keynes, in Chapter 20, uses capital $P$ to also represent expected profit. Let us now define $\overline{P} = \overline{P}_1 + \overline{P}_2$. $\overline{P}_1$ equals the actual realized price of consumption goods and $\overline{P}_2$ equals the actual realized price of investment goods. Then

$$\overline{P}_1 \geq w(dN_1/dO_1) \quad \text{and}$$

$$\overline{P}_2 \geq w(dN_2/dO_2)$$

Then $Y = P_0 = \overline{P}_1O_1 + \overline{P}_2O_2 = C + I$.

Further, $Y = P^* + wN$, where $P^*$ equals actual realized profits received. If $P^* = P$ at $Y = D = Z$ where $e_o = 0$, then the full employment level of output is obtained. Similarly if the expected price, $p$, equals the actual price, $P$, at $e_o$, then $Y = D = Z$. There will be no involuntary unemployment. Finally,

$I = P_2O_2 = (1 - b)Y = (P^* + wN_2)$,

and

(9)
\[ C = P_1O_1 = bY = (P^* + wN_1), \]
so that \[ \bar{P}_1O_1 + \bar{P}_2O_2 = 1 + C = Y = (1 - b)Y + bY. \] (10)

From page 209 and Chapter 21, Keynes integrates monetary considerations. Thus,
\[ M_1V = Y = \bar{P}O, \]
where \( M = M_1 + M_2 = L_1(Y) + L_2(r) \). From Chapters 20 and 21, \( D = MV = PO \). Under what conditions will \( D = Y \), or \( PO = \bar{P}O \), or \( M_1V = MV \), where \( M \) is supply of money? Keynes gives a masterful answer:

"In a static society or in a society...[where] no one feels any uncertainty about the future rate of interest the Liquidity Function, \( L_2 \), or the propensity to hoard will always be zero in equilibrium. Hence, in equilibrium \( M_2 = 0 \) and \( M = M_1 \)...it is a great fault in the Quantity Theory that it does not distinguish between changes in prices which are a function of changes in output and those which are a function of changes in the wage unit. The explanation of this omission is...to be found in the assumption that there is no propensity to hoard and that there is always full employment. For in this case, \( O \) being constant and \( M_2 \) being zero, it follows...if \( V \) is constant...that both the wage unit and price level will be directly proportional to the quantity of money." (Keynes, p. 209)

Note that Keynes asks the reader to read Chapter 21 (Keynes, p. 208). But Chapter 20 is a prerequisite to Chapter 21. Further, given Keynes' footnote on page 25, in Chapter 3 of the GT, one could argue that Chapter 20 is also a prerequisite to a full understanding of Chapter 3 of the GT. For a full and complete understanding of Keynes' Theory of Effective Demand, any reader of the GT should read Chapter 20. Without the logical restraints of Chapter 20, the result has been a Tower of Babel of conflicting interpretations of "What Keynes Meant to Say" arguments.

Keynes' point is that if one assumes only the existence of risk (hard or reliable probabilities) and no uncertainty (soft or unreliable probabilities), then only a transactions demand for money exists and the distinction between SRE of consumer goods (risk) and the LRE of investment goods (uncertainty) is eliminated. There is then no difference in the nature of consumer and investment goods. All expectations are SRE, which leads all the analysis to be framed in terms of risk (Probability Distributions which converge in the long run to their means) so that all long run expectations are realized. Then \( D = MV = PO = Y = \bar{P}O = M_1V = Z = e_0 = 0 \). Say's Law holds.

On pages 137-138, Keynes defines that \( Q_n \) = the expected yield from an asset at time \( s \), \( d_s = \) the present value of 1 pound deferred \( s \) years at the current rate of interest, \( r \). Then \( \Sigma Q_n d_s \) = the expected (demand) price of the investment. The (supply) price is then \( \bar{P}_2 = \Sigma Q_n d_s \) or \( \bar{P}_2 = Q/r \), where \( Q = \Sigma Q_n \) and \( r \) replaced \( d_s \). Then \( r = Q/\bar{P}_2 = m.e.c. \). Given that the supply of money \( M = M_1 + M_2 \) = the demand for money \( L_1(Y) + L_2(r) \), then \( L_2(r) = M - L_1(Y) \).

Define \( k \) to be the constant proportion of \( Y \) needed for transaction balances. Then \( L_1(Y) = kY \) and \( L_2(r) = M - kY \). This analysis is obtained directly from pages 199-202 of the GT.

Now let us make Meade's simplifying assumption "that the ratio between the value of non-liquid assets held and the amount of idle money held is a function of the rate of interest" (Meade, 1937, p. 105). Let \( \bar{K} \) equal the existing fixed amount of actual capital, or stock of investment goods, in the short run; then \( P_2\bar{K} \) equals the current value of existing non-liquid assets (sum of all investment goods). Then
\[ P_2\bar{K}/(M - kY) = L(r). \] (11a)

Keynes would have added
\[ P_2\bar{K}/(M - kD) = L(r). \] (11b)
It is D, and not Y, that goes into equation (11a), given Keynes' discussion starting on page 299 and ending on page 304. Thus,
"Let us write MV=D where M is the quantity of money, V its income-velocity (This definition differing in the minor respects indicated above from the usual definition) and D the effective demand." (Keynes, p. 304)

Keynes' "indicated above" refers to the following:
"The ratio between the quantity of effective demand and the quantity of money closely corresponds to what is often called the 'income velocity of money'; except that effective demand corresponds to the income the expectation of which has set production moving, not to the actually realized income, and to gross, not net, income". (Keynes, p. 299)

Thus, from Keynes' discussion on pages 208-209, maximum output and employment obtain if MV=Y=PO=D=pO=Z at e_o=o if all expected results are realized, including LRE concerning D_3=p_6O.

Then, if expected LRE are obtained, D=Y and (11a) is a simplified result of Keynes' general result that Y≤D.

4 Keynes' Analysis of Aggregate Demand (D) and Aggregate Supply (Z)

In order to examine Keynes' use of D-Z analysis, define six different D functions, D_1-D_6, and six different Z functions, Z_1-Z_6. There will be six different expected prices, p_1-p_6. Define p_6, D_6 and Z_6 to give the maximal amount of output and employment in the short run. This will occur where e_o=o. There is no problem of insufficient aggregate demand or involuntary unemployment. It is a full-employment equilibrium. However, it is only one of a number of multiple equilibriums. Every other p, D, and Z combination also represents an equilibrium, but these are unemployment equilibriums.

In Figure 1^6, the reader will find these six possible equilibria, labeled A, B, C, D, E, and F. Suppose, under decreasing returns to labor, that D_4 was the expected result. N_4 amount of labor will be hired. Note that this is less than the full employment level of output at N_6. N_6-N_4 = the amount of involuntary unemployment, a natural rate of voluntary unemployment being defined to exist at point F.

Suppose that the actual result is Y=D_2. Then total employment will decrease and the amount of involuntary unemployment will increase. Unless something occurs to change producers expectations about future profits and sales, future employment will remain at D_3. On the other hand, suppose Y=D_5 was the actual result obtained. Then total employment will increase from N_3 to N_5. Involuntary unemployment will equal the difference between N_5 and N_6. Now let Y_6=D_6 (=Z_6) at expected price p_6. Here all involuntary unemployment is eliminated. The analysis of the constant returns to labor model, in Figure 2, is practically identical, except there is movement up and down a coincident D, Z, Y line. The a.s.c. becomes vertical at e_o=o. This requires that mpc+mip = 1 = mpc+mps, so that mpi=mps, the marginal propensity to invest equals the marginal propensity to save. All SRE and LRE are being realized temporarily and there is full employment at maximum GNP.

The reader should now move down to the lower half of Figure 1, which is the aggregate labor market. Note the symmetry involved. Keynes works with p_6O-N space when examining the aggregated market for nominal output. The aggregated labor market is in w_f-N space. Note that the involuntary unemployment appears as a disequilibrium in w_f-N space. The unemployment equilibrium is obtained in p_6O-N space (or D_3Z_6Y_6-N space). Given the neoclassical insistence that the demand for labor in the aggregate is a decreasing function only
Fig. 1
of the real wage, while the supply of labor in the aggregate is an increasing function of the real wage, or \( D_N = F(w/p) \) and \( S_N = G(w/p) \), by definition this is the only logical outcome.

Given that Pigou defined the labor market in such terms, Keynes went along with the "rules of the game", although strenuously objecting to them in his appendix to Chapter 19 of the GT. On the other hand, given that 75-80% of short run total costs are the wage and salary costs of labor, at the microlevel, each firm, not observing the interacting feedback effects on actual total sales of reducing the level of total employment in the economy as a whole, is probably correct to concentrate on reducing wage costs from their partial view. A social undepletable, detrimental externality is operating. Keynes' aggregated analysis shows that the whole is more than just the sum of the parts. The market for nominal aggregated output, in \( p_wO-N \) space, incorporates SRE and LRE. The fact that LRE are not being realized is what explains the unemployment equilibrium in \( p_wO-N \) space and the disequilibrium in \( w_p-N \) space. Since the demand for inputs is derived from the demand for output, merely reducing \( w \), the money wage, will have no effect unless money wage cuts could be shown to increase long run physical durable investment, based on LRE, which can't be checked and are discontinuous in the face of technological advance and innovation, which creates the problem of sunk capital, "cast in concrete", which can quickly become obsolete. It would have to be shown that money wage cuts, ceteris paribus, will reduce the high rates of discount applied to the series of expected returns from the use of long-lived durable capital by producers. If this can't be done by the private sector, then \( e_w = 0 \) point will rarely, if ever, be reached. Of course, Keynes' point is that since consumption expenditure and investment expenditure are positively related, while savings and consumption expenditure are inversely related, any increase in current savings must lead to a decrease in current consumption, a fall in sales, a rise in inventories, and a decrease in planned future investment.

Similarly any decrease in savings leads to an increase in consumption expenditure and an increase in planned (expected) investment. The only case where a negative relationship exists between consumption and investment expenditure is if a country is on the boundary of its Production Possibilities Frontier, where all resources are being used in some way or another. All resources are fully employed. Only in this special case would \( mpc + mpi = 1 = mps + mpc \). This would be the same point defined by a D, Z, Y combination at \( e_w = 0 \). Say's Law thus amounts to the claim that all resources are fully employed, so that \( mpc = 1 \) or \( mpc + mpi = 1 \). Thus, if \( mpc \) falls, \( mpi \) increases. If one assumes, a priori, that all resources are fully employed, then of course there can never be any involuntary unemployment or business cycle. Given Keynes' general equilibrium condition, \( w/p = MP_/Mps + mpi \), money wage cuts can only work if \( mpc + mpi = 1 \). But then you are already at full employment!

5 Keynes and Kahn-Robinson, Harrod and Hicks

Consider Figure 3. It juxtaposes Keynes' actual GT model with the interpretations of the GT made by Kahn-Robinson, Harrod and Hicks in a chart similar to the equational structure set out by Young (1987, p. 29) with some minor differences.

We are now in a position to examine Keynes' criticisms of Harrod and Hicks. Keynes' criticisms will revolve around the complete failure of Hicks and Harrod to integrate explicit D and Z functions into their models, as well as to differentiate between SRE (\( D_1-C \)) and LRE (\( D_2-I \)). Keynes unfortunately was far too mild and tepid in his comments. He appears to be "bending over backward" seeking to avoid making critical comments.

In a letter to Hicks (CWJMK, 14; pp 79-80), Keynes, verbally makes it quite clear that in the case of investment goods, it is expected investment, \( D_I \) and expected income, \( D \), and not current realized investment, I, and actual income, \( Y \), which are of primary importance. Hicks never understood this point.
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Figure 3. The structure of the Keynes, Kahn-Robinson, Harrod and Hicks models

($Y =$ realized income; $C =$ actual consumption; $I =$ actual investment; $\bar{K} =$ capital stock; $E =$ expectations; $SRE =$ short run expectations; $LRE =$ long run expectations; $S =$ aggregate savings; $LP =$ liquidity preference; $M =$ quantity of money; $i =$ interest rate; $\bar{P} =$ actual price; $p =$ expected price; $P =$ expected profit).

In a letter to Harrod (CWJMK, pp. 84-86), Keynes states that "You don't mention effective demand", $D = D_1 + D_2$. Keynes then states to Harrod that "You do not show how (author's underscore), in conditions of full employment, which I should now like to define as the limiting case in which the supply of output schedule ceases to be elastic, my theory merges in the orthodox theory". Harrod failed to grasp this fundamental point.

Keynes is referring to the case where $Y = Z = D$ at $e_0 = 0$, where $mpc + mpi = 1 = mpc + mps$, so that $mpi = mps$ and Say's Law holds in describing an economy operating on the boundary of its PPF curve at full employment. Keynes' general theory covers this case and all other cases where $Y < D = Z$, where $e_0 > 0$ so that $mpc + mpi < 1 = mpc + mps$.

Finally, Keynes stated that he had already considered, and rejected, Hicks' interpretation. This means that Keynes regarded Hicks' IS-LM model as being mathematically and economically inferior to Keynes' own $D, Z, Y, D = Z$ a.s.c. analysis in $p_wO-N$ and $w_f-N$ spaces, because it did not incorporate SRE and LRE into the formal model.

For example, Hicks' IS-LM approach can't possibly distinguish SRE from LRE. Thus, "At one time, I tried the equations as you have done with I (current realized income) in all of them. The objection to this is that it overemphasizes current income. In the case of the inducement to invest expected income...is the relevant variable" (CWJMK, 14, pp. 79-80). Keynes needed to come right out and say "you have missed the boat".

6 A Discussion of Meade's Mathematical Model

We begin by defining the variables in Meade's model.

Let $p_x =$ the actual price of capital goods $= w(dN_u/dx)$, \hspace{1cm} (12)
where $x =$ output of capital goods $= \Omega(N_u)$, where $N_u =$ capital goods employment.

Let $p_y =$ the actual price of consumption goods $= w(dN_y/dy)$, \hspace{1cm} (13)
where $y =$ output of consumption goods $= \psi(N_y)$, where $N_y =$ consumption goods employment.

Let $I =$ total realized aggregate income $= xp_x + yp_y$, \hspace{1cm} (14)
Let \( I = P + wN \) equal total variable costs where
\[ P = \text{amount of realized economic profit}. \]  
(15)

Let \( N = N_x + N_y \) equal total aggregate employment
\[ v = (1-s)I \]  
(16)

\[ x_p = sI, \]  
(17)

where \( s = \text{marginal propensity to save} = \text{mps and} \) \( 1-s = \text{mpc} = \text{marginal propensity to consume}, \) so that \( s(1-s) = 1 = \text{mps+mpc}. \)

Let \( i = E(P)/P_x \), where \( i = \text{rate of interest}, \)
\[ P_x = \text{supply price and} \ E(P) = \text{yield expected in each future year}. \]  
(18)

Let \( p_x K = (M - kl) = L(i), \) where \( L = \text{realized current income}, \)
\[ K = \text{the stock of fixed capital goods in the short run,} M = \text{total supply of money,} \ L = \text{the demand for money function, and the left hand side of (19) is the ratio between the value of non-liquid assets and the amount of "idle" money.} \]  
(19)

Meade combines these eight equations with the assumption of a short run analysis under perfectly competitive conditions, where the capital stock is held fixed and technology is given and unchanging. The money wage, \( w \), is uniform in both the consumption and capital goods sectors. Diminishing returns to labor is assumed in both sectors. In other words, the same set of assumptions made by Pigou in TU and Keynes in the GT.

In some ways Meade’s model is inferior to Keynes’, although superior to Hicks’ or Harrod’s. First, Meade never aggregates his model into a one-sector version. Second, Meade has failed to integrate expectations into the microfoundations of his model, although he later, in an ad hoc manner, defines \( \Pi = [P/E(P)] \cdot \frac{dE(P)}{dP} = \text{the percentage rise in expected profits divided by the percentage rise in present profits, which causes the rise in expected profits} \) (Meade, 1937, p. 106; author’s underscore). Note that Meade’s expectational approach is inferior to Keynes’ original formulation. For Keynes, profit expectations can be positive, even if present profits are constant or declining. This means increases in employment can occur without any counteracting increase in the rate of interest, with its negative feedback effect on the quantity demanded of investment, as the transactions demand for money schedule shifts to the right as in the case of increased realized profits necessitating an increased transaction demand for money balances. Meade’s representation of Keynes’ GT model is only valid if all expected results are realized in an optimal fashion. Then \( Y = D = Z \) and Keynes’ model is identical, except for notation, with Meade’s system of equations (12), (13), (14), (15), (16), (17), (18), and (19). The reader is asked to see Figure 4.

Next, Meade is unable to generate the general equilibrium condition for full employment, \( w/p = MPW/(mpc+mpi). \) This condition establishes aggregate involuntary unemployment to be a disequilibrium that can never be eliminated until the underemployment equilibrium in the aggregate nominal goods market is eliminated by increased spending, leading to optimization at \( e_n = 0, \) where \( mpc + mpi = 1 = mpc + mps (mpi = mps). \) Note that the underemployment equilibrium, in the aggregate nominal goods market, is stable. Unless LRE can be changed, the private economy will be in stable equilibrium at less than full employment. Again, see Figures 1 and 2 above. Thus, just as Keynes pointed out, it will be impossible, for labor as a whole, to increase employment by revising their money wage bargains downward with employers as long as \( mpc + mpi < 1. \) In fact, it is mathematically impossible for labor, as a whole, to cut its money wage, since this would violate the rationality postulate of optimizing laborers/consumers, given by the necessary and sufficient first and second order conditions for an optimum. If the workers cut their money wages when \( mpc + mpi < 1, \) this causes a violation of the necessary first order condition for an optimum, given by \( Z, \) since \( dZ/dN = 1 \) must always hold. When \( mpc + mpi < 1, \) then \( MPW/(mpc + mpi) \) is increasing. This requires the ratio \( w/p \) to increase also.
Employment is thus a function of the real wage, \((w/p)\), as well as mpc, mpi, and mps. Since mpi is a function of LRE, which are uncertain, not risky, uncertainty of the future leads to an increase in inactive balances, an increase in LP, a decrease in spending, and a rise in involuntary unemployment. Only if \(M_2=0\), which means that there is no uncertainty about the future, or the future rate of interest, will there be no difference between SRE and LRE. Then SRE = LRE, all expectations are realized, so \(p=\bar{P}\) and \(Y=\bar{P}O=DO=pO=Z\) at \(e_0=0\). You have long run full employment.

Keynes' criticisms of Pigou's one independent variable approach to the aggregate labor market in his appendix to Chapter 19, where all the mathematical analysis is balanced on \(D_N=f(w/p)\) in Pigou's TU, has been misinterpreted as an attack on mathematical economics exposition in general. This is incorrect. It is an attack on an analysis which is incomplete and special. Keynes correctly generalized Pigou's TU.

Finally, Keynes "had no criticisms to suggest" (Young, 1987, p. 34) to Meade because Meade's attempt at dealing with expectations was superior when compared to the analysis of Hicks and Harrod. What caught Keynes' attention was the result reported by Meade in section 4 of Part III of his paper. All of Meade's Marshallian elasticities \(\epsilon_i\), \(\epsilon_m\), \(\epsilon_w\), and \(\epsilon_e\) incorporate the elasticity \(\Pi\) about profit expectations:

"This is what we should expect. A fall in the proportion of income saved will increase expenditure on consumption; and there will therefore be an increase in total expenditure and in employment, unless investment falls by as much as consumption increases. But investment will fall only if there is a rise in the rate of interest, and the rate of interest will rise only if there is a decrease in the supply of 'idle' money. But, with a given total supply of money, the supply of 'idle' money will decrease only if there is an increase in total expenditure, causing an increased demand for money to finance current transactions. The rate of interest cannot, therefore, rise sufficiently to diminish investment by as much as expenditure on consumption has increased" (Meade, 1937, p. 104).
This is a more involved and difficult way of stating that mpc+mpi<1=mpc+mps. If mpc+mpi=1, then "investment will fall as much as consumption increases", and you are in a Say's Law world, operating on the boundary of the PPF. But if "the rate of interest cannot, therefore, rise sufficiently to diminish investment by as much as expenditure on consumption has increased" due to optimistic expectations shifting the investment demand schedule to the right, then mpc+mpi<1. Aggregate output (consumption and investment) must, therefore, not be constant at the full employment level of output, where e1=0. Output can be increased, which means that the relation between consumption and investment spending is positive, not negative, which means the economy has some level of involuntary unemployment.

Meade certainly must be recognized for coming fairly close to duplicating a good deal of Keynes' technical analysis and apparatus. His contribution is vastly superior to that of Kahn-Robinson, Hicks or Harrod.

However, in the intervening years between 1936 and 1987, Meade, like Kahn and Robinson (see Brady 1995), appears to have created his own myth about Keynes and himself.

"It is interesting to note at this point that Meade recollects his paper being turned down by Keynes for publication in the Economic Journal (EJ) because of lack of space and other considerations. According to Meade, Keynes wrote to him saying that while he agreed with it and 'that it was a true representation of the General Theory', he was sorry that he couldn't publish it since not only was the EJ 'inundated and absolutely full of commentary on the General Theory', but that he was already publishing a simplified account of the General Theory in the EJ, which Meade recalls Keynes said was by Reddaway. Meade now thinks that Keynes may have just been trying to put him off at the time, since Reddaway's review of the General Theory came out in the Economic Record of June 1936 (see Chapter 2). In any event, no written rejection of Keynes addressed to Meade is to be found in Meade's correspondence.

More significant, however, is Meade's recollection of the development of his conference paper. According to him, he 'wrote something like it in about 1934' in draft form. He also recalls that 'When the General Theory came out, I wrote it up...I had in my mind [this idea of writing a simple model of determination of employment]...before the General Theory was published'. In Meade's view, 'the macro-model [in Economic Analysis and Policy] is not basically different in structure...I don't think that the thought behind that [Economic Analysis and Policy] differs in any respect whatsoever from the thought behind this ["A Simplified Model of Mr. Keynes' System"]). Ah. (Young, 1987, pp. 37-38)

The reader can turn to Figure 4 and see for himself that Meade's model is a "true representation" only in the simplified case that expected results are always realized. Since Keynes' integration of expectations into value theory and his creation of an aggregated macroscopic analysis of expectations is what the GT is all about, we can be reasonably certain that Keynes never wrote what Meade claims he did. Nor is it likely that Meade ever submitted the paper to Keynes for publication in the Economic Journal. Meade's claims, like those of Joan Robinson, can not be taken seriously, but seem to have "taken" among economists.

7 Patinkin and the GT

In 1965, Patinkin appears to have independently rediscovered some aspects of Keynes' original analysis that he found in Keynes' verbal discussions in Chapters 2, 3, 6, and 19 of the
GT. Unfortunately, Patinkin overlooked the technical, analytic model developed in Chapters 20 and 21 of the GT. Patinkin is correct that it is a deficiency in aggregate demand in the commodity (goods) market that causes, or leads to, involuntary unemployment in the labor market, which must appear, due to the rules of the neoclassical model, as a disequilibrium (in the labor market) (Patinkin 1965, pp. 318-322). The stable underemployment equilibrium appears in the commodity market, or nominal commodity market, for Keynes. Again, see my Figure 1 and 2.

Yet Patinkin can't explain why this occurs. The reasons why Patinkin can not explain this phenomenon are fundamental. First, for Patinkin, there is no distinction between risk (SRE) and uncertainty (LRE). Thus, there is no real difference between the market behavior of producers purchasing capital goods and consumers purchasing consumption goods. In both cases, reliable and accurate probability distributions provide information about the mean and standard deviations of all alternative courses of action. In effect, all decisions are reversible. Thus, since producer and consumer goods are approximately the same, in terms of making choices to buy or sell, there is no need for a two-sector model which carefully distinguishes between the two types of goods and their differing supply elasticities. It is the investment goods' portion of the commodity goods market that causes the problem of involuntary unemployment due to the simple fact that I is usually less than D. The result will be suboptimal plant sizes and suboptimal levels of equipment, machinery, tools, robots, etc., for the labor force to work with. Patinkin's "labor off their supply of labor curve" analysis needs to be revised in a future edition. It is the uncertainty of LRE, leading to k-D and the later heavy discounting of the flow of returns to capital goods that explains the demand deficiency. All the action takes place, then, in the commodity market in that part labelled investment goods. The labor market is theoretically passive. There is no simultaneous solution in a set of general equilibrium equations.

Second, Patinkin, in his use of a quantitative, empirical "real balance effect", forgets that a mathematical model can only incorporate qualitative effects. Thus, a mathematical economics model of perfectly (or purely) competitive input and output markets can only yield purely qualitative results from a theoretical analysis of the model's variables. Patinkin's real balance effect involves an explicit empirical, quantitative claim about the distribution of money and financial assets between creditors and debtors. Only if we assume explicitly that, in a purely competitive economy, the majority of consumer-producers are net creditors, will the real balance "effect" start to close the involuntary unemployment gap in that portion of the goods market dealing with consumption goods. It will have no impact on investment goods.

On the other hand, if the number of net debtors is greater than the number of net creditors, the involuntary unemployment gap will increase. Finally, in the case where the number of net debtors equals the number of net creditors, no reduction will occur in involuntary unemployment. There will be no real balance effect whatever.

Patinkin appears to recognize this objection (Patinkin, 1965, pp. 285-288, 336-337), but goes ahead anyway in constructing a spurious theoretical addition to the standard purely or perfectly competitive model. Patinkin, like Pigou (Pigou, 1943), confused the qualitative nature of a mathematical model with the quantitative nature of a statistical or econometric model. His new variable, real balances, is not qualitative in nature, it is quantitative.

Nor is Patinkin "more Keynesian than Keynes" (Patinkin, 1965, p. 340), for the simple reason that a real balance "effect", even if there is one, will have no effect on the investment demand portion of the aggregate commodity or goods market, since LRE, involving uncertainty, are very different from the SRE, involving risk, that impact producers and consumers, in the consumption goods portion of the commodity market.

Throughout his exposition, Patinkin blinds himself to Keynes' analysis of the nature of the investment goods problem by his use of the word "uncertainty" which really means risk, as
measured by a well defined stable and known probability distribution's standard deviation, (Patinkin, 1965, pp. 117-136, 82-88, 336, 450-456) whereas Keynes means that there is a lack of a sufficient amount of information in order to even identify what the specific, relevant probability distribution is with any high degree of reliability. Of course, this is measured by Keynes' \( w \) variable contained in his "conventional coefficient of risk and weight" decision rule \( c \) (Keynes, 1921, p. 315), where \( a \leq w \leq 1 \). \( w \) is a measure of the completeness of the information that plausibly could be obtained, now and in the near future, on alternative courses of action. Thus, if the total existing amount of information on physical capital goods is small, relative to some credible expectation of relevant future additions to the available amount of information, \( w < 1 \). But, to specify a well defined probability distribution, that is accurate, requires a \( w = 1 \). Since Patinkin has built his analysis of investment decision making on the spurious belief that there is available accurate probability distributions that effectively deal with technological change and its impacts on the mean return-standard deviation outcomes producers can expect from purchasing durable long-lived capital goods, it is simply impossible for him to understand Keynes' argument. To his credit, Patinkin did not overlook section 4 of Chapter 15 of the GT on pp. 208-209, which was Keynes' introduction to Chapter 21. Unfortunately, having overlooked Chapter 20, Patinkin failed to grasp the difference between \( Y = \bar{P} O \) and \( D = pO \), where \( Y = \) current realized income (expenditure) and \( D = \) expected future income (Keynes' supply side).

Contrary to Patinkin, this part of the GT has nothing to do with an "earlier discussion on p. 199" by Keynes in the GT (Patinkin, p. 638). Keynes' argument concerning the Quantity Theory of Money on p. 209 is correct, because, from Keynes' view of the neoclassical perspective, \( M_1 \) (\( M \)) is strictly a transaction demand for money operating in an economy according to Say's Law. In such a world, there is only money, which is neutral, consumption goods, investment goods, and bonds, newly issued by firms, which are exchanged for the savings of consumers and used to purchase investment goods. There is no government or public sector. There is no organized financial asset markets trading and dealing in old issues. Bond sales are permanent. Patinkin disputes Keynes' view about the operational meaning of Say's Law (Identity) for neoclassical, or classical, economists (Patinkin, 1965, pp. 364-65), arguing that Say's Law meant that, in the long run, wage, price and interest rate adjustments would move the economy to a long run optimum. Patinkin's interpretation of how neoclassicals (classicals) defined Say's Law does not in any way impact Keynes' results, since, unless \( mpc + mpi = 1 \), the necessary general equilibrium condition, \( w/p = M/P (mpc + mpi) \) can't be obtained even in the long run. Unless \( mpc = 1 \) or \( mpc + mpi = 1 \), there will continue to be involuntary unemployment in the long run, for a private, purely or perfectly competitive economy.

Finally, Patinkin's claim that the Quantity Theory of money does not depend on the assumption that \( L_2 = 0 \) in equilibrium is incorrect if the entire set of Keynes' assumptions are examined (Patinkin, 1965, p. 257, 638). Patinkin, while pointing out one of the additional assumptions Keynes required, "that there is always full employment", forgets to add Keynes' additional provisos, that therefore "\( O \) being constant" and "if we can take \( V \) also as constant" (Keynes, 1936, p. 209). Then "both the wage unit and the price level will be directly proportional to the quantity of money" (Keynes, 1936, p. 209). This is correct.

In conclusion, consider the following statement of Patinkin:

"Another point which is essentially at issue ... is whether Keynes was completely clear in his mind about the proper analytical relationship between the labor market and the commodity market. And here let me simply point out that this is a problem which is still being thrashed out at the frontiers of macroeconomic theory. I struggled with it in Chapter 13 of my Money, Interest, and Prices (1956, 1965). It was also one of the major problems that
subsequently concerned Axel Leijonhufvud in his influential work *On Keynesian Economics and the Economics of Keynes* (1968). It is the central problem of Robert Barro and Herschel Grossman's recent monograph on *Money, Employment and Inflation* (1976). And in a sense it is also a basic aspect of the 'microeconomic foundations of macroeconomics' problem that has concerned such writers as Edmund S. Phelps (1970), Robert E. Lucas (1973), Donald F. Gordon (1976), and others in their ongoing studies of the Phillips curve and its implications for aggregate supply" (Patinkin, 1978, 579-80).

Now consider the following statement by King:

"To emphasize the importance I attach to the influence of expectations on aggregate demand, the current essay does not concern the role of expectations on the aggregate supply side. Despite their importance, my sense is that economic theories of aggregate supply -- of either a new Keynesian or new Classical sort -- are simply too underdeveloped for economists to evaluate the relative importance of expectations and other mechanisms in explaining aggregate supply behavior" (King, 1993, p. 70).

Contrary to both Patinkin and King, Keynes succeeded 60 years ago in integrating expectations into both the microeconomic foundations of macroeconomics and the aggregate supply side using the analytic tools available at that time in the 1930's.

8 Barro, Grossman, and the GT

Barro and Grossman (BG) generalize Patinkin to include, not only deficient demand, but also excess demand. They have re-obtained some of Keynes' results from Chapters 20 and 21 of the GT. Unfortunately, BG are at a loss when it comes to explaining why a demand deficiency exists, which they model as a quantity constraint. This is due, as with Patinkin, to their working with a one sector aggregated goods or commodities market, which thus excludes from the start problems of uncertainty and LRE which are unique to the investment goods sector, and which account for the sub optimal size and amount of plant and equipment, respectively.

In order to obtain Keynes' general equilibrium condition for optimality in both the goods and labor market

\[ w/p = MPV/(mpc + mpi), \]

where mpc + mpi is < mpc + mps = 1 for the demand deficiency case, you must have a two-sector model. LRE involve uncertainty concerning which probability distribution to apply, discontinuities, and irreversibilities concerning decision choices about the quantity of investment goods to use, due to constant technological change and innovation, creating sunk capital and obsolete capital stock problems for producers. This leads to very high rates of discount in net present value calculations applied to investment decisions, the creation of an investment gap, which then leads to involuntary unemployment, demand deficiency, and decreasing sales. On the other hand, the SRE about consumer goods decisions (ordering, stocking, cutting or raising prices to clear the market) involve risk, continuity, and reversible actions. Thus, BG's claim that "It should be clear that the incorporation of investment and a market for securities would alter none of the conclusions advanced in this paper" (p. 84, ft.7) would only be true if they assume that there are no differences in the analysis of consumption goods and investment goods. Of course, this means that LRE = SRE, uncertainty = risk, decisions are reversible in a continuous world of well defined probability distributions. BG commit the same error as Patinkin, incorporating a real balance effect, which necessarily requires an empirical, quantitative distribution specification. None is provided in their paper. Next, while the labor market must be in disequilibrium, due to the one independent variable
specification of the labor market, note that, contrary to BG, the goods market (or nominal goods market) is always at some equilibrium due to the multiple equilibrium nature of this market. Again, see figures 1 and 2 above. The only way to reach the D₈-Z₈ combination at \( e₈=0 \) requires a change in LRE, leading to a shift from, say, D₉-Z₉ to Z₈-D₈. Real balance effects can have no effect on LRE since shifts in D, Z, and Y represent changes in aggregate demand and supply, not changes in the quantity demanded and supplied, involving movement up or down a fixed, invariant curve, ceteris paribus.

Finally, Keynes' demand for labor function can be re-written, under the supposition that it is no longer necessary to have a one independent variable version in order to compare with Pigou's 1933 TU model in Chapters 9-10 of Part II. Then

\[
D₈ = F(w/p, mp, mpi, mps, D, Z).
\]

Only at \( D₈ = Z₈ = Y \) at \( e₈ = 0 \) in Figure 1, where mp+mpi=mpc+mp=m, does Keynes' \( D₈ \) function simplify to \( D₈ = F(w/p) \) only. Keynes spent much of his discussion in the appendix to Chapter 19 of the GT on this very point. Thus, the claims of BG (see p. 82, p. 82, ft. 2 and 3, p. 93) about Keynes accepting the neoclassical \( D₈ \) function are simply false. Keynes' analysis of causality, running from the nominal goods market to the market for labor with the \( D₈ \) being a function of total spending-savings at the margin, generalizes the very special neoclassical case.

The reader should note that BG's model involves dividing through by \( P \), the money price of commodities, whereas Keynes and Pigou divide through by \( w \), the wage unit, uniform in both sectors.

9 Darity - Cottrell, Rappoport and Meade's Model

Darity-Cottrell (1987) and Rappoport (1992) both re-examined Meade's model, with Rappoport presenting an aggregated one-sector version. Both papers correctly find that Meade's inclusion of an elasticity of expected profits variable and his analysis of its effect on the investment sector, in a formal mathematical analysis, was much closer to what they think Keynes had in mind when juxtaposed with the Hicks-Harrod IS-LM model. Unfortunately, both Darity-Cottrell and Rappoport have completely overlooked Keynes' own model, which I have demonstrated to be superior to Meade's model, in Chapters 10, 20-21 of the GT.

Second, there is no logical slip in Meade's model. Meade, following Keynes, is setting out to demonstrate the existence of involuntary unemployment in a perfectly, or purely competitive "Say's Law" world. In such a world, there is no financial market, "there is no interest-bearing paper asset for them to trade" (Darity-Cottrell, 1987, p. 213). Besides inside money, consumer goods, and investment goods of different qualities, vintages and durability, there is only a market for newly issued bonds. Consumers use their savings to buy the new bonds, which they can't trade, from the owners of investment goods, who then use the savings to buy more investment goods. Thus, in introducing a market where "equities may be traded at any time on a stock exchange" (Darity-Cottrell, p. 213) and making the substitutions for Meade's mec and liquidity preference equations on p. 214 of their article, they violate one of the rules of a Say's Law world, which requires the "identity" specified by Meade.

Finally, Darity-Cottrell's belief that "Meade in 1937 could have provided a superior starting point for the algebraic formalization of the General Theory." (Darity-Cottrell, 1987, p. 221), is simply false. This is the same mistake made from Samuelson (1946) to Skidelsky (1992). See my discussion above.

Rappoport (1992) constructed a one-sector version of Meade's model. The supply-side portion of this model, like Meade's, is identical to the original analysis of Keynes in Chapter 20 of the GT, once it is assumed that \( D=Z=Y \). For instance, Rappoport's \( Y=f(N), f'>0, f'<0 \) becomes, in Keynes' notation, \( 0=\Omega(N), \Omega'(N)>0, \Omega''(N)<0 \). Rappoport's \( p=f(N) \) is Keynes' \( p=\omega(N) \), Rappoport's \( py=R+wN \) is Keynes' \( Y=\Omega=0=D=p=Z=P+wN \).
(Rappoport, 1992, p. 354). Further, Rappoport's \((\text{wN/py}) = \mu\) is Keynes' 
\(\text{wN/p0} = \text{wN/D} = N/p\circ (1/p)\circ (1/\text{MPN}) (1/\text{APN}) = \text{APN}/\text{MPN}\). Rappoport's \(\hat{\theta} = -\left(\hat{e}^2\right)/\hat{f}\)

is actually the aggregate supply version of Keynes' employment function analysis of Chapter 20 of the GT, which was \([\theta'(N)]^2/\theta(N)\circ \theta'(N)\), which would be written as 
\([\psi'(N)]^2/\psi(N)\circ \psi'(N)\) in the notation of footnote 2 on p. 55 of the GT. See Rappoport, 1992, p. 363.

I have already gone over the defects of Meade's modeling in a section above. If Rappoport had not overlooked Chapters 20 and 21 of the GT, he would not have stated that:

"The discussion of long-term expectations in Chapter 12 of the General Theory has been used more as a source of Keynes' purple prose than of insights to be incorporated in formal models. Yet Keynes' reaction to the initial responses to the General Theory was to reiterate the general message of Chapter 12, emphasizing the importance of uncertainty and the way expectations and the propensity to hoard interact to determine the rate of capital accumulation."

(Rappoport, 1992, p. 357)

Keynes' LRE were already formally incorporated in his Y, D, Z, D=Z model, specifically in the \(D_2\) and \(Z_2\) components where \(p_1\), \(p_2\) and \(p\) are expected prices and \(P\), \(P_1\), and \(P_2\) are expected profits. If LRE are fulfilled, then \(I=D_2=Z_2\) at \(e_\circ=\circ\) and mpc=mps. If LRE are not fulfilled, then \(I<D_2=Z_2\), \(e_\circ>\circ\) and mpc<mps. Thus, Keynes' aggregated model of 1936 is also superior to both the 1987 Darity-Cottrell two-sector version of Meade and the 1992 Rappoport one sector version of Meade.

10 Conclusions

The mathematical model presented by Keynes in 1936 in the GT is superior, in terms of its economic content, incorporation of expectations and mathematical completeness, to any of the mathematical formalizations undertaken by economists who attempted to interpret Keynes. With respect to the models of Harrod and Hicks, Keynes' major point, that Harrod and Hicks failed to incorporate expectations in general and LRE in particular, into their IS-LM model, basing macroeconomic analysis on realized results, and not expected results, is correct. This means that the IS-LM model can't deal with the distinction between SRE and LRE or with LRE.

With respect to Meade, his modeling of the elasticity of profit expectations, his integration of this elasticity into his system of four major elasticities, and his attainment, verbally, of Keynes' general equilibrium condition, that mpc+mpi<mpc+mps=1, met with Keynes' approval.

Patinkin, and later Barro-Grossman, re-discovered portions of Keynes' analysis. They are correct that unemployment must appear in the labor market as a disequilibrium. Further, they correctly surmised that it is the commodity or goods market where the problem arises. Unfortunately, they introduced a non-qualitative, non-mathematical, quantitative, empirical variable into their qualitative models. This illicit variable was called "real balance effect". Once this is removed and the basic logical necessity of differentiating consumer goods from investment goods and SRE from LRE is made, then the multiple equilibrium nature of the goods market becomes clear.

Lerner's argument that "...Mr. Keynes' book has been rendered obsolete by a more careful presentation of its argument at the hands of other writers" (Lerner, 1941, pp. 298) is simple nonsense. As of 1996, no such superior treatment has appeared.
At best, neoclassical models can only represent special cases of Keynes' general model, using such special assumptions as the future is risky, but not uncertain, SRE are the same as LRE, the supply-side elasticities of consumption and investment goods are identical, investment and consumption expenditure is inversely related, the economy is always at the NRU (natural rate of unemployment), the economy is always on the boundary of the Production Possibilities Frontier, mpc = 1 or mpc + mpi = mpc + mps = 1, there is no asset demand for money, money is neutral, there are no inactive balances. Many other such special assumptions could be added to the above list. In any case, once these special assumptions are made, Keynes' Theory merges with the neoclassical. Whenever, which will be very rare, the neoclassical assumptions turn out to be sound representations of the real world, any results obtained will be no different from those obtained using Keynes' model of the GT in 1936.

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Notes

For instance, Weintraub claims that "unfortunately, there is today no accepted view of what it was that Keynes actually accomplished... For better or worse, questions about 'what Keynes really meant' stand apart from the creation of rigorous, analytically tractable, and sufficiently rich microfoundations structures" (Weintraub, ER, 1979, 38-39).

Lerner claims that "the essence of the Keynesian idea is that...prices do not fall. Keynes wanted to show that it was just not rigidity...so he tried to give other reasons...I think these other reasons are unnecessary and confusing... An adequate mechanism for the reduction of prices...would cure any unemployment automatically" (Colander and Landreth, 1996, p. 95).

Samuelson claims practically the same thing as Lerner (Colander and Landreth, 1996, pp. 160-162). Tobin states:

"Seeking to win the game on his opponent's home field, Keynes pretended to be assuming pure competition in all markets. But his insights regarding labor markets implicitly recognized that wages are administered or negotiated prices, and for that reason alone are not perfectly flexible, not prices set in impersonal auction markets. His product markets, however, remained Marshallian" (Tobin, 1993, pp. 56-57).

Tobin has overlooked Chapter 20 and 21 of the GT. In Keynes' model, the labor market is purely or perfectly competitive, the same as Pigou's assumptions in 1933.

Chapple (1995) makes a number of unsubstantiated claims such as:

"Three Systems' contains a simple integrated general equilibrium treatment of goods and money markets and links with marginalist short-period pricing theory in a fashion analytically superior to anything that Keynes or the Stockholm school had published by 1934" (Chapple, 1995, p. 533);

and,

"Thus by integrating the treatment of goods and money markets with marginalist price theory, and by incorporating into this the theory of effective demand and 'quasi-equilibrium', it would seem that in most respects Kalecki anticipates the Keynesian model" (Chapple, 1995, 535).

The reader should compare the model that Chapple claims for Kalecki (Chapple, 1995, pp. 529-30) with both Keynes' and Meade's actual models. None of the variables in the model explicitly incorporate expectations.

Reddaway's review of Keynes' GT is a commendable effort, although there are a few errors and a major omission. For instance, in discussing Keynes' analysis "of risk and the varying estimates made" (p. 103), Reddaway overlooked that Keynes had already explicitly stated in detail on page 68 of the GT that "we deduct an allowance to cover it (risk) from the expected yield in calculating the marginal efficiency of capital" (Reddaway, p. 104). Keynes repeats himself on page 240 of the GT.

Reddaway did not understand Keynes' critical comments, regarding the misuse of formal mathematical analysis in economics, contained in the GT. They are directed at Pigou's 1933 TU
analysis of the labor market, where one, and only one, independent variable is allowed, the real wage. The statement that "Mr. Keynes, quite rightly in my opinion, depreciates the spurious air of exactness introduced by too much mathematics. But in his endeavor to describe the system without this sort of shorthand he has tended to obscure the fact that the determination is mutual" (Reddaway, p. 107) leads Reddaway, like Paul Samuelson, to omit from consideration the discussion of one-half of Keynes' mathematical model, the supply-side expectational analysis and microeconomic foundations contained in Chapters 20 and 21 of the GT. I present a brief improved and rewritten version of Reddaway's model

\begin{align*}
(1) \ S &= f(Y), \text{ where } Y = C + I + PO \\
(2) \ I &= g(D_x), \text{ where } D = D_1 + D_2 + pO \\
(3) \ I &= S \geq D, \text{ where } I = S = D_1 \text{ at } \epsilon_1 = \epsilon_2 = 0 \text{ is optimal} \\
(4) \ M &= L_1(Y) + L_0(t).
\end{align*}

In July 1946, Paul Samuelson (PS) published his "Lord Keynes and The General Theory". Given the results presented in this paper, I must take issue with some of the statements made by Samuelson. These statements have been passed down to succeeding generations of economists and become the conventional wisdom of the economics profession. They have contributed to creating a severe bias against Keynes as a theoretician. For example, G. Mankiw states 20 times in his interview article with Snowden and Vane (Snowden and Vane, 1995, p. 49, 51, 53, 54, 57, 59, 60, and 61) that Keynes, and the "old" Keynesian position built on Keynes, had no microeconomic foundations or a "good" theory of aggregate supply. Thus, for one example,

"I don't think that Keynes was as worried about building his model based on micro foundations as we are today. Joan Robinson was building the microeconomics that would later prove to be very useful for addressing the macroeconomics of Keynes, but Keynes, not having read Robert Lucas yet, wasn't worried about building the microeconomics of aggregate supply (laughter)". (Snowden and Vane, 1995, p. 54)

Of course, the microfoundations for Keynes' theory of aggregate supply can be found in completely developed fashion, in Chapters 20 and 21 of the GT. Obviously, Mankiw, Snowden and Vane never read that far in the GT. The omission of Chapter 20 is fatal. Samuelson makes seven unsubstantiated claims. I will present these seven claims and then demonstrate that all seven are false.

Claim number one:

"My rebellion against its pretensions would have been complete except for an uneasy realization that I did not at all understand what it was about. And I think I am giving away no secrets when I solemnly aver - upon the basis of vivid personal recollection - that no one else in Cambridge, Massachusetts, really knew what it was about for some 12 to 18 months after its publication. Indeed, until the appearance of the mathematical models of Meade, Lange, Hicks, and Harrod there is reason to believe that Keynes himself did not truly understand his own analysis."

(Samuelson, 1946, p. 188)

Claim number two:

"Herein lies the secret of the General Theory. It is a badly written book, poorly organized; any layman who, beguiled by the author's previous reputation, bought the book was cheated of his 5 shillings. It is not well suited for classroom use. It is arrogant, bad-tempered, polemical, and not overly-generous in its acknowledgements. It abounds in mares' nests and confusions: [involuntary unemployment], [wage units], the equality of savings and investment, [the timing of the multiplier,] [interactions of marginal efficiency upon the rate of interest] [forced savings,] [own rates of interest], and many others. In it the Keynesian system stands out indistinguishably, as if the author were hardly aware of its existence or cognizant of its properties; and certainly he is at his worst when expounding its relations to its predecessors. Flashes of insight and intuition intersperse tedious algebra. An awkward definition suddenly gives way to an unforgettable cadenza. When it finally is mastered, we find its analysis to be obvious and at the same time new. In short, it is a work of genius."

(Samuelson, p. 190)

Claim number three:

"It is not unlikely that future historians of economic thought will conclude that the very obscurity and polemical character of the General Theory ultimately served to maximize its long-run influence. Possibly such an analyst will place it in the first rank of theoretical classics along with the work of Smith, Cournot, and Walras. Certainly,
these four books together encompass most of what is vital in the field of economic theory; and only the first is by any standards easy reading or even accessible to the intelligent layman."

(Samuelson, p. 190)

Claim number four:

"In any case, it bears repeating that the General Theory is an obscure book so that would-be anti-Keynesians must assume their position largely on credit unless they are willing to put in a great deal of work and run the risk of seduction in the process. The General Theory resembles the random notes over a period of years of a gifted man who in his youth gained the whip hand over his publishers by virtue of the acclaim and fortune resulting from the success of his Economic Consequences of the Peace. Like Joyce's Finnegans Wake, the General Theory is much in need of a companion volume providing a 'skeleton key' and guide to its contents: warning the young and innocent away from Book I (especially the difficult Chapter 3) and on to Books III, IV, and VI. Certainly in its present state, the book does not get itself read from one year to another even by the sympathetic teacher and scholar."

(Samuelson, p. 190)

Claim number five:

"No author can complete a survey of Keynesian economics without indulging in that favorite indoor guessing game: Wherein lies the essential contribution of the General Theory and its distinguishing characteristic from the classic writings? Some consider its novelty to lie in the treatment of the demand for money, in its liquidity-preference emphasis. Others single out the treatment of expectations. I cannot agree. According to recent trends of thought, the interest rate is less important than Keynes himself believed; therefore, liquidity preference (which itself explains part of the lack of importance of the interest rate, but only part) cannot be of such crucial significance. As for expectations, the General Theory is brilliant in calling attention to their importance and in suggesting many of the central features of uncertainty and speculation. It paves the way for a theory of expectations, but it hardly provides one."

(Samuelson, p. 191)

Claim number six:

"In all candor, therefore, it is necessary to point out certain limitations -- one might almost say weaknesses were they not so intrinsically linked with his genius -- in Keynes' thought. Perhaps because he was exposed to economics too young, or perhaps because he arrived at maturity in the sullfying backwash of Marshall's influence upon economic theory -- for whatever reason, Keynes seems never to have had any genuine interest in pure economic theory. It is remarkable that so active a brain would have failed to make any contribution to value theory; and yet except for his discussion of index numbers in Volume I of the Treatise and for a few remarks concerning 'user cost,' which are novel at best only in terminology and emphasis, he seems to have left no mark on pure value theory."

(Samuelson, p. 195)

Claim number seven:

"From Marshall's early influence, no doubt, stems Keynes' antipathy toward the use of mathematical symbols, an antipathy which already appears, surprisingly considering its technical subject, in the early pages of the Treatise on Probability. In view of the fact that mathematical economists were later to make some of the most important contributions to Keynesian economics, his comments on them in the General Theory and in the Marshall and Edgeworth biographies merit rereading. Moreover, there is reason to believe that Keynes' thinking remained fuzzy on one important analytical matter throughout all his days: the relationship between 'identity' and functional (or equilibrium-schedule) equality; between 'virtual' and observable movements; between causality and concomitance; between tautology and hypothesis. Somewhere, I believe in the 1923 Tract, he already falls into the same analytic confusion with respect to the identity of supply and demand for foreign exchange which was later to be his stumbling block with respect to the identity of saving and investment."

(Samuelson, op.cit.)
All of PS’s incorrect statements occur due to PS’s ignorance of Keynes’ mathematical model in Chapters 20 and 21 of the GT. PS’s claim number one is simply false. The reader need only compare the Hicks, Harrod, and Meade models with that of Keynes’ in my figures 1-4. PS’s claim number two is simply false. The “Keynesian system” and “its properties” stand out clearly in Chapters 20 and 21. The “tedious algebra” and “awkward definition(s)” integrate expectations into the microfoundations of the theory of the firm. PS’s claim number three is false. How could a very obscurely written book of a polemical nature attain a first ranking in theoretical classics? One need only reread the Mankiw interview and/or the analysis of Lucas, which follows, to see that the current long run influence of the GT is nil.

PS’s claim number four is, again, the usual case of an economist’s reading Chapter 3 alone and ignoring the analysis of Chapters 20 and 21. PS’s claim number five follows from his claims one and four. PS does not understand the difference between $C+I+Y=\bar{Y}$ and $D_1+D_2=D=p_0$, combined with $Z_t+Z_w=Z=P+wN$, where $p$ is an expected price, $P$ expected profit and $\bar{P}$ an actual price. Keynes provided a fully integrated micro-macro theory of decision making and expectations by firms-industries. This was done by Keynes in Chapters 20 and 21 of the GT.

PS’s claim number six and seven provided him with the logical rationale as to why Keynes did not present a mathematical model of his theory in the GT. Supposedly, this was due to the fact that Keynes was not interested in “pure economic theory”, or “pure value theory” because of the supposedly “outfitting backwash of Marshall’s influence upon economic theory”. This assessment is false. Contrary to Samuelson, Marshall understood the limitations of mathematical analysis as well as its potency. Both Pigou’s 1933 *TU* and Meade’s 1937 *RES* article are built directly on Marshall’s “outfitting” elasticity approach to mathematical economics, an approach that Samuelson was, and is, ignorant of.

All seven of PS’s claims, when combined with similar unsupported claims made by Joan Robinson and Richard Kahn, form the foundation of the current assessment of Keynes’ work. This foundation is simply incorrect. Thus, it is imperative that economists reconsider Keynes’ contribution from the perspective of pure theory. Without such a reconsideration, we will be treated to more of the following types of conversations exemplified by the Klameter-Lucas exchange of 1982:

"You told me that reading Keynes is impossible for you. Has that anything to do with language?
I was just kidding. But I don’t like that book. It’s not a very congenial book to read.
Some people find it the best economic book ever written.
I really find that amazing. I find it carelessly written, not especially gracefully written, sometimes dishonestly written. I don’t like the bullying tone. I don’t like the sort of British aristocratic stuff. He’s got this great idea for solving index number problems – we’ll measure everything in units of workers. Workers are obviously homogeneous: that solves the index number problem [laughter]. It would be impossible for an American to come up with the idea that workers are homogeneous. It’s a ridiculous idea. What the hell was he thinking about? But it’s written loosely enough that if I liked the book, I probably would find enough quotes in there that I could use to motivate arguments. But there’s so much arbitrariness in that book.”

(Klameter, 1984, pp. 50-51)

In this statement, Lucas appears to be ignorant of the theoretical fact that in the input portion of the purely or perfectly competitive theory of the firm, each worker is a perfect substitute for every other worker, that is, every worker is homogeneous. This gives a completely elastic $S_N$ curve for the firm in input space. Keynes’ discussions on pp. 40-43 of the GT represent a vast improvement in using this standard perfectly or purely competitive assumption for solving the problem of aggregating employment from the level of the firm in a two-sector model to an aggregated one sector model. This is not “a ridiculous idea”. It is not “written loosely enough”; it is not “arbitrary”. In fact, it is just basic principles of microeconomics.

Skidelsky’s Chapters 15 and 16, section III, reflect the unfortunate influence upon him of what appears to have been private discussions with Richard Kahn and Joan Robinson. Skidelsky’s claims that Kahn “was a much better mathematician than most of those in Keynes’ circle”, that “he recalled Keynes himself as being a poor mathematician by 1927”, and that Kahn’s “ability to think mathematically was to be of crucial help to Keynes in the early 1930’s” (Skidelsky, p. 288) are not supported by any evidence.

Skidelsky’s own view of Keynes’ *Fundamental Equations* is the first warning sign that Skidelsky will be totally incapable of maintaining any critical scholarly position when he attempts to discuss “what Keynes meant” in his Chapters 15 and 16, whenever the issue arises concerning
mathematical modeling and the GT. According to Skidelsky, Books III and IV of the TM are "full of dreadfuly convoluted sentences. They are also replete with pages of dismal algebra (the Fundamental Equations) in which one identity is transformed into another with the help of an increasing pile of symbols until the reader has forgotten where he started from or what they mean." (Skidelsky, p. 317). Contrary to Skidelsky, the Fundamental Equations represent an aggregated version of purely or perfectly competitive input and output markets. Since these equations are derived from the underlying micro, optimizing behavior of firms and workers, they can't be identities. The symbols involved are standard symbols that any mathematically literate economist in 1930 or 1992, would have been able to follow quite easily.

I have already demonstrated that Keynes' mathematics model of Chapters 10, 20-21 of the GT is superior to those of Meade, Hicks, and Harrod. Thus, contrary to Skidelsky, his claim that the "difference between Keynes and Hicks is that "the difference is between expectations being talked about (as Keynes did in Chapter 12) and being put into the formula as Hicks did" (Skidelsky, p. 615) is false. First, Hicks did not incorporate expectations into his model. Second, the real difference is between Keynes' correct modeling of LRE, by means of \( D_2 = Z_2 \geq 1 \), up to the point where \( e = 0 \), when by definition all expectations are realized at \( 1 = \overline{D}_2 = \overline{Z}_2 = \overline{O}_4 \). Thus, Hicks' "formula" is, when compared to Keynes', a very inferior formulation. No wonder, then, that "In his long letter to Hicks there is not a single word of praise...no appreciation at all of the very singular feat Hicks had accomplished" (Skidelsky, p. 615).

Why should Keynes praise an economist who has abysmally failed to incorporate the distinction between expected and realized income into his model, an inferior model at that. Why should Keynes praise an economist who had no inkling of what Keynes was doing in Chapters 20 and 21? Contrary to Skidelsky (p. 575). Keynes' assessment of Hick's Theory of Wages (1932) as "'the serious and careful work of an unoriginal but competent mind" is accurate and correct.

Finally, Keynes did not allow "his mathematics to rust away because...he was not interested in 'mathematising' the social sciences", (Skidelsky, p. 412) as Skidelsky claims. Nor did Keynes' "skepticism about the use of mathematics in economics grew rather than diminished with age..." (Skidelsky, p. 412). Keynes' mathematical skills can be studied and appreciated by any economist who is willing to read Chapters 20 and 21 of the GT. Such an economist can then compare Keynes' modeling approach with that of Pigou on pp. 88-106 of TU or of Meade. Finally, Keynes' comparison-constrain in the appendix to Chapter 19 of the GT between himself and Pigou will be self-evident to those who have made the effort to analyze Keynes' "Cambridge elasticities" in Chapters 20 and 21 of the GT.

In Figures 1 and 2, it was shown how Keynes' one-sector aggregated mathematical model appears in a graphical format. There is a one-to-one, onto mapping, or isomorphism, between Keynes' \( (D_w-N) \) \( \rightarrow \) \( O \) space and the aggregate labor market \( w-N \) space. However, an important point made by Keynes was that the supply side elasticities in the consumption and investment goods industries are usually different, with the investment goods industry employment having a more elastic response than the consumption goods industry, which has a less elastic response in employment, with respect to changes in effective aggregate demand. Thus, an initial increase in investment spending, be it either private or public, would have the greatest impact in reducing unemployment. However, to analyze this graphically would require four additional nominal goods market-aggregate labor market diagrams, two each for the case of diminishing returns, \( D_w-N_1 \) and \( D_w-N_2 \) and then two more for the case of constant returns to labor.

In a letter from Dennis Robertson to Alvin Hansen, we find the following claim:

"My own complaint has always been that Keynes cannot make up his mind whether his verbal labels apply [only to equilibrium positions] or also [to disequilibrium ones.] In these pages he seems to be concerned only with the former, so that while his D is explicitly defined (Keynes, 1936, p. 25) as expected sales proceeds, it does not much matter if we interpret it, as you do, as actual sales proceeds, alias money income. But later on (Keynes, 1936, p. 78) we suddenly find a distinction between effective demand (= expected income) and actual income not merely drawn but declared to be "vital for causal analysis"; and we are left gasping with bewilderment as to what is the relation between this "vital" distinction and the central analysis of pp. 24-30 (Keynes, 1936), where it is never mentioned (23 September 1953).

(Mizen and Presley, 1995, p. 648)

Robertson's fundamental error, like Hansen's, was to view Chapter 3 of the GT and "pp. 24-30" as "central analysis". When, in fact, what Keynes had written was a brief introductory outline which was to serve as a guide to what Keynes would do in "later chapters". It is in Chapters 19, 20, and 21 of
the GT, and in the appendix to Chapter 19, that Keynes provides the technical analysis backing up and filling in the sketch of Chapter 3.

Robertson repeats this mistake again in 1955 (Robertson, 1955, pp. 474-78). This error also afflicts H. Johnson's mathematical appendix to Robertson's paper. Three errors are contained in Johnson's appendix. First, the price term p in Keynes' D-Z analysis stands for the expected general price level, not the actual general price level. Second, the aggregate supply function is Z = P = wN, where P = expected profit. Z is not the aggregate supply curve. Thus, contrary to Johnson, Z ≠ Johnson's px, because px = D = expected aggregate demand = expected total revenue from future expected sales. D = Z only at expected maximums. Johnson never states this qualification. Thus, \( \frac{dZ}{dN} = \frac{dP}{dN} = w(1) \). \( \frac{d^2Z}{dN^2} = \frac{d^2P}{dN^2} = 0(2) \). The mathematical analysis provided by Johnson on page 478 of Robertson's paper is not "his results as regards Z" (Robertson, p. 475). The result is unnecessary to establish the convexity of the Z = D locus. See figures 3 and 4. There is no need to take a third derivative. Johnson's errors are repeated in Davidson and Smolensky (1964, p. 125) and Vickers (1987, pp. 90-93).

There has been a great debate over Chapter 2 of the GT concerning Keynes' definitions of involuntary unemployment. Just as the debate over "What Keynes Meant" in Chapter 3 of the GT is resolved through a study of Chapters 20 and 21, so also is the debate over Chapter 2 resolved. The two definitions Keynes gives are equivalent. One definition applies to \( w_p \)-N space and the other definition applies to \( p_O(D_p) \)-N space.

Assume that all expected results are realized. Then \( c = 0 \). At this point, the a.s.c. becomes vertical in \( p_O-N \) space. You obtain a full employment equilibrium in \( w_p-N \) space as a result of the outcome in the nominal goods market. If \( c > 0 \) in \( p_O-N \) space, you obtain an unemployment equilibrium in \( p_O-N \) space and an unemployment disequilibrium in \( w_p-N \) space. In \( w_p-N \) space, workers are involuntarily unemployed, since, given "a small rise in the price level, due to an increase in the price of wages goods relative to the money wage, both the aggregate supply of labor...and the aggregate demand for it at the wage would be greater than the existing volume of employment" (Keynes, p. 15). For the case of diminishing returns to labor, analyzed graphically in Figure 1, Keynes is correct when he states that an alternative definition "amounts, however, to the same thing" (Keynes, p. 15) since \( w_p \) in \( w_p-N \) space is the inverse of \( p_O \) in \( p_O-N \) space. Thus, a movement down the aggregate demand for labor curve in \( w_p-N \) space is identical to movement up the a.s.c. in \( p_O(D_p)-N \) space.

Authors such as Darity and Horn (1983), Darity (1985), Darity and Horn (1987-88), Darity and Massero (1981), Lawlor, Darity, and Horn (1987), Smith (1985) and McCombie (1987-88) have overlooked Keynes' unifying analysis contained in Chapter 20 of the GT. While, for instance, Darity and Horn point out that there is a Chapter 20 in the GT where "Keynes worked explicitly with Cambridge-styled elasticities" (Darity and Horn, 1983, p. 725), and that full employment occurs at \( c = 0 \), they ignore the rest of Keynes' Chapter 20 analysis wherein Keynes integrated \( p_O(D_p)-N \) space and \( w_p-N \) space. Garrison (1987-88) correctly points out this omission out by citing Keynes' verbal analysis from p. 283-284 of the GT, but does not attempt to point out the more powerful mathematical results contained in footnotes 1 and 2 on p. 283 or in the formula (4-1-6): \( w_p = \frac{1}{c} \). The contributions of Darity and his various co-authors should thus be viewed as supplements or complements to Keynes' analysis, when additional independent variables are added to the demand and supply of labor functions in an aggregated analysis. However, their analysis is not Keynes'. Keynes, in order to make an effective contrast with Pigou, chose to keep the \( D_w \) and \( S_w \) functions of one independent variable, the real wage. By keeping the supply side the same, Keynes was able to show the incompleteness of neoclassical economics, which had no aggregate demand side nor analysis of expectations incorporated into the standard theory of the firm.

7 A. Hansen never grasped the revolutionary aspect of Keynes' integration of expectations into value theory, although he suspected it to be the case. Hansen never grasped the difference between Y and D-Z. Thus,

"The current orthodox theory - represented for example by Pigou - has so fully elaborated the theory of underemployment equilibrium under conditions of cost rigidities and monopolistic control of supply, that it is only necessary here to make reference thereto" (Hansen, 1936, p. 680, also p. 678).

Contrary to Hansen, the theory of effective demand and involuntary unemployment has absolutely nothing to do with wage, price, or cost rigidities or imperfect competition or monopolistic competition. The same error was made by R. Kahn, J. Robinson, and Kalecki. P. Samuelson exemplifies this ubiquitous error:

"Had Keynes begun his first few chapters with the simple statement that he found it realistic to assume that modern capitalistic societies had money wage rates that were
sticky and resistant to downward movements, most of his insights would have remained just as valid." (Samuelson, in Lekachman, 1964, p. 332)

If this were the case, then no innovation was produced by Keynes in the GT. Of course, this is not the case. From the purely or perfectly competitive model, the necessary first order condition for a profit maximum is \(w/p = (MP_N)\). Let us follow Samuelson’s recommendation and introduce imperfect competition. The necessary first order condition for a maximum is \((w/p)(1-1/c_0) = MP_N(1+1/c_0)\). Monopoly (monopsony) and oligopoly (oligopsony) in the output (input) markets result in a smaller level of real aggregate output and hence a smaller level of employment. Thus, unemployment will be higher than levels occurring in a competitive economy. It appears that this is what Pigou, Robertson, Hawtrey, Hicks, Samuelson, Kahn, Robinson, Kalecki, etc., interpreted Keynes as trying to say, in an unrigorous, problematical way, that this higher unemployment was what Keynes meant by involuntary unemployment. However, this is incorrect. Keynes made it very clear, in Chapter 2 and throughout the GT, that he was working with an aggregated version of a perfectly/purely competitive world. Keynes then showed how he had succeeded in generalizing the neoclassical model by his contribution to pure value theory - distinguishing and integrating SRE of consumption goods and LRE of investment goods. In perfect competition, only SRE about the prices and wages of final output and input factor costs are known with certainty. LRE about the future price movements of long lived investment goods, distinct from consumption goods, or final output, and based on ambiguous vague, distorted, unclear, imperfect, and incomplete information, are totally ignored. Given that \(mpc(SRE) + mps(LRE) = mpc(SRE) + mps(SRE)\), the optimality condition in a perfectly competitive model is \(w/p = MP_N/(mpc+mps)\).

Involuntary unemployment is due to a deficiency in the effective aggregate demand for investment, or producer, goods so \(mpc+mps<1\). Only if we assume the special neoclassical case that \(mpc+mps=1\) can \(w/p = MP_N\) obtained. In general, it is not obtained.

Once Keynes had succeeded in specifying the pure theory of Effective Demand, he would have had no objection to generalizing the condition to account for imperfect competition. We obtain \((w/p)(1-1/c_0) = MP_N(1+1/c_0)/(mpc+mps)\).

Both Kahn and Robinson never understood Keynes’ pure theory of Effective Demand. Blinded by their own joint work on imperfect competition, they were incapable of seeing Keynes’ point.

Keynes regarded decisions to purchase particular types of capital goods as being irrevocable, irreversible, and fixed. LeRoy (1983) correctly demonstrated that Keynes’ pure theory of investment, in both the TM and GT, can only be represented by a two sector model, with non-shiftable (fixed) capital, of temporary (shifting) general equilibrium. LeRoy gives the correct specification of Keynes’ demand-supply theory of investment, at the bottom of p. 414 and in footnote 8, at the bottom of p. 415, in terms of the demand prices of capital goods, in his paper.

Upon this framework of fixed non-shiftable capital goods discussed in Chapter 11 of the GT, Keynes generalized his theory of investment decision making in Chapter 12 and in Chapter 17, on pp. 239-241 of the GT, to include all assets.

Using Keynes’ weight of evidence index, \(0 \leq w \leq 1\), from his Treatise on Probability, the difference between consumption and investment goods expectations can be traced out. Consumption goods expectations (SRE) are risky. The \(w\) of consumption goods expectations \(w_c\) is far greater than the \(w\) of investment goods expectations, \(w_i\). The total amount of evidence, past and present, about the expected prices and profits from the projected sales of consumption goods, \(D_p=p_iO_i\) and \(Z=wP_i+wN_i\), is great while the completeness of the evidence is high. The probability distributions concerning \(D_p\) and \(P_i\) are at worst, only mildly inaccurate or unreliable. SRE can be represented as risky, with linear additive probabilities summing to one. For investment goods, the total amount of evidence, about the expected prices and profits from the projected sales of investment goods, \(D_p=p_fP_f\) and \(N_p=0\), is small. Likewise, the completeness of the evidence given an environment of technological change and innovation that continually generates new relevant information over time, is, at any point in time, very low. Essentially, LRE are uncertain, with non linear, non-additive probabilities the rule.

Keynes felt that the LRE of \(D_p\) were based on a very low \(w\). In contrast, neoclassical economists make no distinction between expectations concerning \(D_p\) and \(D_f\). They are both risky. Hence, \(w_c = w_f = 1\). On the other hand, post Keynesians argue that while \(w_c\) is high, \(w_f = 0\). For post Keynesians, no probabilities can be calculated or ranked ordinarily as the producer investor is always dealing with what Shackel called unique situations that are organic and dissimilar; hence, non-repeatable. No inductive knowledge can be gathered from the study of past and present decision situations which would allow some degree of similarity to be recognized as a pattern. Some other form of non-probabilistic analysis must be used.
for instance, Shackle's theory of potential surprise. Probability is not the guide to life for post Keynesians.

Keynes concluded that firms dealt with the uncertainty of the future concerning $D_2$ and $P_2$ by a process of constantly attempting to delay, or by spreading out, decisions to purchase new producer goods or build new plants. Even if purchased or built, such projects would have been evaluated by applying inordinately high rates of discount when calculating the expected future return. This will lead to insufficient capital equipment and smaller than optimal plant sizes.

For Keynes, capital is not a homogeneous putty that can effortlessly be remolded into different forms over time. Physical capital goods are cast in concrete. The decision is usually irreversible. It can't be re-cast and reused in a different industry if the original demand for its output disappoints the expectations of the buyer. Significant time must pass before the equipment is used up in a sub-optimal manner.

In Chapter 17 of the GT, Keynes generalized his model of Chapters 10, 11, 20, and 21 to include all assets, not merely physical consumption and investment goods. Thus, Keynes deals with the full range of assets - money, bonds, stocks, land, commodity futures, etc. Let $Q =$ yield, $c =$ carrying cost, $l =$ liquidity premium, and $a =$ appreciation or depreciation, with all variables being measured in terms of itself (own rates). Then expected yield $= \lambda Q$ and expected appreciation or depreciation $= \Pi a$, where $\lambda$ is the probability of the yield and $\Pi$ is the probability of the appreciation or depreciation.

Then the expected total return $ETR = \lambda Qte^\Pi a + c$. Now think of $I$, the liquidity premium, as a measure of the degree of confidence an investor has in his estimated expected return or yield. Thus, $I = g(w)$ As $w$ decreases, $I$ will increase, while if $w$ increases, $I$ will decrease. Then a weighted $ETR = ETR[(2w)/(1+w)]=WETR$, where $[(2w)/(1+w)]$ is the independent weight of evidence portion of Keynes' conventional coefficient of risk and weight (see Brady and Lee, 1991; Brady, 1993 and Brady 1994c). Then, as $w$ approaches 1, $I$ approaches 0. You would be left with the ETR, which deals only with risk. On the other hand, as $w \to 0$, $I$ approaches some very large premium for those assets that are very unreliable and which are judged to be based on vague or ambiguous evidence. Thus, the expected monetary value rule and subjective expected utility rule turns out to be special cases of Keynes' general rule.

The reader should note that some notational difficulties are involved in Keynes' comparison-contrast of his supply side analysis with that of Pigou's in the appendix to Chapter 19 of the GT. Pigou's two-sector model is expressed in the notation $X + y = \mathcal{O}(X)$, where the variable $X =$ wage goods employment, the variable $y =$ non-wage goods employment and $\mathcal{O}(X)$ is a function. Keynes' two sector model is $N_1 + N_2 = N$, or $\Sigma N_0 = N_1 + N_2 = N$, where the variable $N_0 =$ consumption goods employment, the variable $N_1 =$ investment goods employment, and $N$ is a variable representing aggregate employment. Clearly, $X + y = \mathcal{O}(N)$ and $N_1 + N_2 = N$ can't be compared unless Keynes' model is reduced from three variables to two.

Now, in his one-sector version, Pigou redefined $X$ to represent total aggregate employment. Thus, $X = \mathcal{O}(N)$ reduces to $X$, and Keynes' one-sector model, using $N$, can be directly compared with Pigou's from the supply side. Of course, Keynes sought to demonstrate to Pigou how his model could be "reinterpreted" so as to allow for a multiplier effect from government expenditure on public works projects. In order to do this, Keynes must redefine $N$, in a special two sector version of his GT model, to equal Pigou's wage goods. Now $X + y = \mathcal{O}(X)$ and $N + N_0 = \mathcal{O}(N)$ are comparable. Of course, this model was only for the purposes of the appendix to Chapter 19 of the GT. Notice also that the exact same problem arises if it is argued that Keynes' interpretation of Pigou should have been $X + y = \mathcal{O}(y)$. Again, $X + y = \mathcal{O}(y)$ is not comparable with $N_1 + N_2 = N$.

Keynes states that Pigou's "function $\mathcal{O}$ is...a function of what I have called above the employment multiplier $k$" (Keynes, 1936, p. 273). Of course, Keynes is referring back to his discussion on p. 115 of the GT, where $\Delta N = k \Delta N$. Keynes defines $\Delta N$ to equal the change in total employment given a change in $\Delta N_1$ of investment goods employment. Keynes then states that $\Delta X = k' \Delta N$ so that $\mathcal{O}(X) = 1 + 1/k''$ (Keynes, GT, p. 273). This is the best that Keynes could do to fix up Pigou's model. Keynes thus redefined $N$ to stand for consumption goods employment so that a change in investment goods (Pigou's non-wage goods) employment has a multiplier effect on consumption goods employment (Pigou's wage, or final, goods). Given Pigou's definition of $X + y = \mathcal{O}(X)$, this is the only way a multiplier effect can be generated within Pigou's model that is mathematically correct. Since Keynes also assumed, for simplicity, that $k = k' \Delta N$, Pigou's $\mathcal{O}(X)$ can also be interpreted as an investment expenditure multiplier $k$. Keynes can then reinterpret Pigou's $\mathcal{O}$ as a "propensity to consume wage goods" (GT, p. 273), i.e., as a sort of consumption function. Of course, Pigou rejected all of this. Pigou's argument was that there is full employment at the given or fixed real wage $w/p = F(X)$. Look at
my Figure 1. Pigou has only one D function, \( D_a \), and one Z function, \( Z_e \). Since expected results are always realized, \( Y \) always equals \( D_a = Z_e \) at \( e = 0 \), unless there is an excessively high money wage, in which case you move back to the left along \( AD_a \). Pigou's model concerns changes in the quantity demanded of labor, not the changes in demand that Keynes correctly showed was the heart of the problem, resulting from changes in expectations. For Pigou, the economy is on the boundary of the PPF unless, due to excessive money wages, employers have to charge excessive prices, which, ceteris paribus, reduces quantity demanded of output and the quantity demanded of labor. Pigou's entire analysis is, ultimately, a function of one variable, the real wage. Thus, his elasticity analysis is redundant.

Keynes' consumption function and multiplier analysis follow directly from Keynes' letter of September 20, 1931, to Richard Kahn. Kahn, in his reply, demonstrated that he has no idea what Keynes is talking about:

"When resources are fully employed, the supply schedule for goods as a whole is inelastic. Thus any new factor can only have reactions of two kinds - to cause changes in the proportionate division of output between consumption and investment goods and to bid up (or down) the remuneration of the factors of production. But, if there is already unemployment, there is a third reaction possible, namely changes in total output" (Keynes, 1973, p. 373-374).

Of course, Keynes has just given an excellent verbal discussion of what it means to be on the boundary of the Production Possibilities Frontier versus being in the interior. Full employment is a boundary phenomenon, subject to some natural rate of unemployment. However, once you are in the interior, you have involuntary unemployment. Further, "we have 'long-period unemployment', i.e., an equilibrium position short of full employment" (Keynes, ibid., p. 374).

Of course, it is precisely this kind of unemployment that neoclassicals reject. The issue is most clearly seen in the Keynes-Hawtrey correspondence. Keynes states:

"There is less than full employment if, the propensity to consume being assumed unchanged, an increase in investment will cause an increase in consumption. As against this the normal assumption of the classical theory is that an increase in investment will involve a decrease in consumption" (CWJMK, 1973, p. 26, Vol. 14).

Hawtrey responded:

"That seems to be equivalent to saying...that an increase in output is possible. If that is the test, then you mean by full employment precisely what other people mean by it, and it is not true that the classical economists assume it (EXCEPT OF COURSE AS A STATIC PRINCIPLE)" (author's capitalization, CWJMK, Vol. 14, 1973, p. 33).

Of course, Hawtrey's "static principles" are properly termed neoclassical economic theory. Given that technological change is being held fixed by both Pigou and Keynes, there can be no dynamic shift outward (or inward) in the PPF. Keynes' theoretical argument rejects the "static principles" except as a special case.

First, we will examine the neoclassical position. Define \( I = \)investment goods on the abscissa (X-axis) and \( C = \)consumption goods on the ordinate (y-axis). Connect both axis with a bowed out concave curve, called the PPF. Then, for every point on the PPF, we have \( Y = C + I \), where \( Y \) is a constant, representing fixed output. Then \( dY = dc + di \) and \( dY/dY = 1 = dc/dY + di/dY \). If \( dc/dY > 0 \), then by definition, \( di/dY < 0 \) and if \( dc/dY = 0 \), then by definition \( di/dY > 0 \). Thus, we have identities or Say's Law, that \( C \) and \( I \) are negatively related.

Keynes rejects this as a special case holding only on the boundary of the PPF. First, define \( C \) to be a positive function of \( Y \), or \( C = bY \), where \( dc/dY = b > 0 \) but less than one. \( dc/dY = b < mp \). Then \( dY = dc + dc = bY + I + (1-b)Y - I = I(1-b). Then dY/dY = 1/(1-b). dY/dY = (1/(1-b)) > 0. Thus, there is a positive relationship between \( Y \) and \( I \). Given that \( C \) is a positive function of \( Y \), then \( C \) and \( I \) are positively related. Thus \( dc/dY = 1/(1-b) \). Then \( b/(1-b) > 0 \), given that \( b < 1 \).

The neoclassical theory that \( C \) and \( I \) are inversely related is a very special case. holding only if an economy is on the boundary of the PPF. Keynes incorporates this special case along with his more general case to establish a general theory.

The reader should also note that, by definition, if the economy is in the interior of the PPF, there will always be unused resources available to generate the multiplier process. Increased saving in the interior merely drives you deeper into recession or depression. Only if an economy is operating on its boundary would increased savings be needed before investment projects were undertaken.

This analysis also demonstrates the confused state of mind of D. Robertson (1938) concerning where the financing of increased investment expenditure is to originate from. Paralleling the real
economy is a monetary-financial economy. By definition, there will exist inactive balances available as long as one is in the interior of the PPF.

The BG model has some very basic deficiencies, visavis Keynes' GT model. I have already mentioned above their failure to differentiate consumer goods from investment goods and hence, LRE from SRE. There is no elasticity analysis provided anywhere in the article. In my opinion, this, alone, would have led to its being rejected from the *Economic Journal* of the 1930's. BG assume that prices are fixed. Thus, changing and/or shifting expectations about prices are excluded from the analysis. It is this failure on the part of BG, to incorporate price expectations, that prevents them from obtaining multiple equilibria in the nominal goods market. Thus, expectations are restricted to real output. In dividing through by a fixed price level of commodities, they ignore the aggregation problem of specifying a correct price index. In the rest of the footnote, I demonstrate the close connections in the mathematics of the BG and K (Keynes) models.

Both models work with the basic short run profit=total revenue-total variable cost approach, which is the exact same approach of Keynes in the TM, Pigou in his TU, Keynes in his GT, and Meade in his 1937 model. For Keynes, \( Q = \text{output}, N = \text{labor}, p = \text{expected price}, \bar{P} = \text{actual price}, O = \Omega(N), O^+ = \Omega^+(N) < O \). For BG, \( y = \text{output}, x = \text{labor}, p = \text{actual price}, y = f(X), F'(X) > 0, F''(X) < 0 \). For K, \( w = \text{money wage rate}; \) for BG, \( w = \text{money wage rate and } w = w/P = \text{real wage rate} \). (The reader should note that BG do not include this step in their paper formally, since it is obvious.) The same rationale holds with my representation of Keynes' actual profits by \( P^* \) combined with K's definition of \( P (P = P_1 + P_2) \) as expected profits. For BG, \( P \) can equal either expected or realized real profits. \( P = P_1 P_2 \).

The reader should note that Keynes works with \( pO \) or \( P O \), nominal terms, due to the aggregation problem he felt was of paramount importance. Obviously, if prices are fixed in Keynes' model, you would get the same results obtained by BG.

For Keynes, \( P = pO \) - \( wN \) or \( P = D - wN \) where \( D = D_1 + D_2 \) or, if \( Z = P + wN \), \( D = Z \). This is the expected supply side results based on the microtheoretical assumption that firms optimize. For Keynes, the actual result is \( P^* = \bar{P} O - wN \), or \( P^* = Y - wN \). If firms successfully optimize, then all expected results are realized and \( D = Z = Y \) at \( e^0 \). For BG, \( \Pi = y^*-wX^0 \), where \( y = \text{real expected sales and } X^0 = \text{expected labor demanded} \). This is the result based on the assumption that firms optimize. For BG, the actual result is \( \Pi = Y - wX^0 \), where \( y = \text{real actual sales and } x^0 = \text{actual demand for labor} \). If firms successfully optimize, then all expected results are realized and \( \Pi = y^*-wX^0 = \Pi = y - wX^0 \). Thus, \( y = y^* \) and \( x^0 = x^0 \). If we add on elasticity conditions, similar to Keynes' \( e^0 \), then BG's result is identical to Keynes' result. Involuntary unemployment, in BG's notation, is the difference between \( x^0 \) and \( x^0 \). The rest of BG's paper adds in an incorrect real balance variable and hence is of no interest.

**References**


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