The Development of Keynes' Theories of Risk in Chapters 26 and 29 of the *Treatise on Probability*

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Keynes made an extensive investigation of Chebyshev's Inequality, the Normal Distribution, the binomial distribution, and the logical and mathematical connections between them in Chapter 29[1] of the *Treatise on Probability* (TP). At the technical level, Keynes made extensive use of calculating formulas for both the variance and standard deviation in Chapter 29. Consider Keynes' analysis on page 387 of the 1973 CWJMK edition (1921, pp. 353-4). Using Keynes' notation but restricting ourselves to only one of the three random variables analyzed by Keynes, define a random variable X which can take on values $x_1, x_2, x_3 ..., x_k$ with probabilities p_1, p_2, p_3, p_k, where $\sum_1^k p_k = 1$, and $\sum_1^k p_k x_k = a =$ the mean. Then

$$\sigma^2 = \text{the variance} = \sum_1^k (x_k - a)^2 p_k$$
$$= \sum_1^k \left(x_k^2 - 2ax_k + a^2\right)p_k$$
$$= \sum_1^k p_k x_k^2 - 2a\Sigma p_k x_k + a^2 \Sigma p_k$$

But $\Sigma p_k x_k^2 = a$, or in modern notation $E[X^2]$, where E = "expected value of". Similarly, $\Sigma p_k x_k = a$, or $E[X]=\mu$. Finally $a^2\Sigma p_k = a^2$, since $\Sigma p_k = 1$. We obtain

$$\sigma^2 \quad = a_1 - 2a^2 + a^2$$

$= a_1 - a^2$, (other results would be $b_1 - b^2$ and $c_1 - c^2$ given $\Sigma q_\lambda = 1$ and $\Sigma q_\lambda y_\lambda = b$, etc.) where $a^2 = \mu^2$, $2a^2 = 2\mu^2$ and $a_1 = E[X^2]$. In modern notation, we have

$$\sigma^2 = E[X^2] - [E(X)]^2.$$

Given Keynes' calculation of the variance above and calculation of the standard deviation later in Chapter 29 of the TP, as well as the fact that his example of R=qpA is a simplified version of the lottery problem of Chapter 29, it is clear that Keynes had a theory of risk which is much more conventional than the theory of 'risk' presented in Chapter 26. In Chapter 26, Keynes writes the word risk as 'risk' and then states that risk "*may be defined in some such way as follows*" (author's underscore). A study of Chapter 29 makes it clear that Keynes would also accept the rules to minimize risk = $qpA^2(nqpA^2)$, $\sqrt{qpA}$, ($\sqrt{nqpA}$) as well as qpA (nqpA).

Keynes obtained his Chapter 26 problem by setting s=1 and B=0 in $s(pA-qB) \pm \propto(A+B)\sqrt{spq}$. This give us $pA \pm \propto A\sqrt{pq}$

This of course is a "modern" confidence interval, where $(A\sqrt{pq})^2$ gives us the variance A^2pq. Keynes' 'risk' formula is a hybrid of the standard deviation and variance. Now let us apply the Chapter 29 analysis to Keynes' example from Chapter 26, where E= the mean = pA, the outcomes were 0 and A, and the probabilities were p and q, where p+q=1. We obtain the following:

$$\sigma^2 = \sum_{k=0}^{k=A} (x_k - pA)^2 p_k = (O - pA)^2 q + (A - pA)^2 p$$

$$=p^2A^2q+pA^2-p^2A^2-p^2A^2+p^3A^2$$

$$=(p^2A^2)q+pA^2-p^2A^2-p^2A^2+p(p^2A^2)$$

$$=(p+q)(p^2A^2)+pA^2-p^2A^2-p^2A^2$$

$$(p+q) = 1 \qquad =1(p^2A^2)+pA^2-p^2A^2-p^2A^2$$

$$=pA^2-p^2A^2$$

$$=p(1-p)A^2$$

$$=pqA^2$$

$$=(qpA)A$$

$$=RA$$

where R=qpA is Keynes' Chapter 26 example. Now Samuelson, very briefly, notes this (1977, p. 47) but, due to his overlooking Chapter 29 of the TP, he ends his analysis precisely where it should have started.

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Note

1. The following additional typographical errors in Chapter 29 are corrected to help readers of the TP avoid confusion. See Brady (1996) for the more serious typographical errors involving "a" instead of the correct "$\propto$".

On page 386 (1973 ed.), 22nd line from the top:

$$\sum_1^{\ell} q = 1 \text{; not } \sum_1 q = 1 .$$

On page 386, 23rd line from the top:

$$\sum_1^m r_\mu z_\mu = c \text{ ; not } \sum_1^m r_\mu r_\mu = c$$

On page 386, 24th line:

$$\sum_1^k p_k x_k^2 = a_1 \text{ ; not } \Sigma p_k k_k = a.$$

On page 386, 24th line:

$= c_1$, etc., not $= c$, etc, (same correction, page 353, 1921).

On page 387, 15th line from the top:

$p_k q_\lambda r_\mu$; not $p_k x_\lambda r_\mu$

On page 390, line 17 from the top:

$$\sqrt{\frac{n(\mu - n)}{\mu}} \text{ ; not } \sqrt{\frac{\mu(\mu - n)}{\mu}}$$

$$\left(\frac{s}{\mu}\right)^{n+1} \text{ ; not } \left(\frac{s}{n}\right)^{n+1}$$

On page 390, 21st line from the top (page 357, 21st line from the top, 1921):

$$\left(\frac{s}{\mu}\right)^{m} \text{ ; not } \left(\frac{s}{m}\right)^{m}$$

On page 390, 22nd line from the top (22nd line from the top, page 357, 1921):

$$\left(\frac{s}{\mu}\right)^{n+1} \text{ ; not } \left(\frac{s}{n}\right)^{n+1}$$

On page 390, 22nd line from the top (22nd line from the top, page 357, 1921):

$$\left(\frac{\mu-s}{\mu-n}\right)^{\mu-n} \text{ ; not } \left(\frac{\mu-s}{\mu-n}\right)^{n-\mu}$$

Finally, the following paragraph on pages 353-354 of the original edition of the TP, 1921, has been omitted from the 1973 CWJMK edition. Keynes originally wrote the following:

"The probability that the sum $x+y+z+\ldots$ will have for its value $x_\kappa+y_\lambda+z_\mu+\ldots$ is $p_\kappa q_\lambda r_\mu\ldots$(provided that the values of x,y,z...are independent) Hence $\Sigma(x_\kappa+y_\lambda+z_m+\ldots a-b-c\ldots)^2$ $p_\kappa q_\lambda r_\mu\ldots$summed for all values of κ, λ, μ is the average expectation for $S(x_\kappa+y_\lambda+z_\mu+\ldots a-b-c-\ldots)^2$."

The Σ sign is missing from the last expression in the 1921 edition. The above paragraph has been replaced by the following in the 1973 CWJMK version:

"Consider the expression $\Sigma(x_\kappa+y_\lambda+z_\mu+\ldots-a-b-c\ldots)^2\, p_\kappa q_\lambda r_\mu\ldots$".

Nowhere has it been noted, by the editors of the CWJMK, that Keynes' original wording has been completely altered at least to this reader's knowledge. If the editors of the CWJMK wish to alter the original writing of J.M. Keynes, they should at least have the courtesy to inform the reader that Keynes' analysis is being changed by putting a footnote on the page(s) where the alteration appears.

References

Brady, M.E., (1996). "Decision Making Under Risk in the *Treatise on Probability*: J.M. Keynes' 'Safety First' Approach", *History of Economics Review*, Winter/Summer 25, 204-209.

Keynes, J.M. (1921). *A Treatise on Probability*, London: Macmillan (AMS Press Reprint, 1979).

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Samuelson, P. (1977). "St. Petersburg Paradoxes: Defanged, Dissected and Historically Described", *Journal of Economic Literature*, 12, 24-55.