The Development of the Theory of Exchange*

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Abstract

The aim of this paper is to provide an outline of the development of the theory of exchange, concentrating on the less well-known development of the formal model which culminated in the contribution of Edgeworth. The importance of exchange, viewed as the central economic problem for the early neoclassical economists, is stressed. Instead of taking a chronological approach, non-utility approaches are first discussed. These include the extension by Walras of Cournot's attempt to model trade between regions, and Whewell's mathematical version of J. S. Mill's international trade analysis, followed by Marshall's diagrammatic version. Jevons's and Walras's utility approaches are then examined, showing the different paths they took from the same basic equations of exchange. After a very brief discussion of Edgeworth, the neglected but valuable contribution of Launhardt, along with the later work of Wickesell, are examined. Emphasis is placed on the similarity of the formal structure of the exchange model used by the various writers. This similarity has been obscured by the different forms of presentation used and the emphasis given to various aspects and results by each investigator.

*I have benefited greatly from many discussions with Denis O'Brien, along with his comments on earlier related papers, during the long gestation period of this paper.
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1 Introduction

A distinguishing feature of the economic analysis of roughly the last third of the 19th century was its emphasis on exchange as the central economic problem. For example, Hicks (following Edgeworth) referred to the early neoclassicals as 'catallactists' in order to emphasise the exchange focus. He stressed that, 'while the classics looked at the economic system primarily from the production angle, the catallactists looked at it primarily from the side of exchange. It was possible, they found, to construct a "vision" of economic life out of the theory of exchange, as the classics had done out of the social product. It was quite a different vision' (1984, p.250). Edgeworth (1925, ii, p.288) summarised the position by suggesting that, 'in pure economics there is only one fundamental theorem, but that is a very difficult one: the theory of bargain in a wide sense'.

There are two primary ingredients of an exchange analysis: one is an appreciation of the principle of reciprocal demand and supply while the second is the concept of demand as a function of relative prices. Early examples of treatments of exchange, including Aristotle, Beccaria, Verri, Courcelle-Seneuil, Turgot, Cantillon, Canard and Isnard, show how little real progress can be made without these two elements combined, despite the useful insights provided. However, the neoclassical economists were not the first to recognise these requirements or attempt to construct a model of exchange based on them. International trade provided an important context in which exchange theories were considered; prime examples include Cournot, J.S. Mill and Whewell. Only Mill and, following him, Whewell combined the two elements successfully.

The great success of the early neoclassical economists was also associated with the fact that they provided a foundation for their exchange model in the form of a utility analysis. This allowed for a deeper treatment of the gains from exchange and the wider consideration of economic welfare. Furthermore, this type of welfare analysis survived the replacement of a cardinal utility concept with an ordinal concept, or the idea of a simple preference ordering. As Hicks (1984, p.252) argued, 'I would therefore maintain that the principal reason for the triumph of catallactics – in its day it was quite a triumph – was nothing to do with socialism or individualism; nor did it even have much to do with the changes that
were then occurring in the "real world". The construction of a powerful economic theory, based on exchange, instead of production and distribution, had always been a possibility. The novelty in the work of the great catallactists is just that they achieved it.³

It is only when the perceived central position of exchange analysis is recognised, along with the place of the principle of utility maximisation as the foundation, that it is possible to have some appreciation of the attitude behind Edgeworth's (1881, p. 12) remark, after discussing the extension of utility analysis to subjects such as production and labour supply, that, "Mécanique Sociale" may one day take her place along with "Mécanique Celeste", throned each upon the double-sided height of one maximum principle, the supreme pinnacle of moral as of physical science . . . the movements of each soul, whether selfishly isolated or linked sympathetically, may continually be realising the maximum energy of pleasure, the Divine love of the universe". Other writers were much more prosaic in their expressions than Edgeworth, but his view nicely encapsulates something of the pioneering spirit of the early neoclassical economists.⁴

The central role of exchange is unfortunately seldom stressed in modern texts or histories of economic analysis, where stress is placed on the idea of a 'marginal revolution' associated with the concept of marginal utility, which of course arises naturally from the first-derivatives needed in a utility maximising approach. The emphasis is such that priority of place is often given to the adjective rather than the noun (in marginal utility), with stress on the introduction of calculus methods, or at least notation.⁵ The context of discussions is typically the derivation of partial equilibrium demand curves, even though such curves hardly ever appeared in the early major works of Jevons, Walras, Edgeworth, Wicksell and even Marshall. It is suggested that this view of the neoclassicals is not helpful and actually creates something of a barrier to obtaining an understanding of their approach.⁶

The aim of this paper is to provide an outline of the development of the theory of exchange. Rather than taking a purely chronological approach, the discussion is divided into three main sections, dealing with non-utility approaches, the introduction of a utility foundation, and finally expositions and extensions.

Section 2 begins by examining the non-utility approaches to exchange, beginning in subsection 2.1 with Cournot's (1927) model of trade between two regions,
involving a single good and dating from 1838: reference is generally made below to the available source, in Cournot's case the translation edited by Fisher, rather than the original date of publication. Both Walras and Marshall rejected this approach and recognised that it could not be extended simply by adding more demand and supply equations; the fundamental concept of reciprocal demand and supply has to be at the heart of any exchange model. As shown in subsection 2.2, Walras took the most direct route while Marshall took Mill as his starting point. Mill's analysis, along with the mathematical model produced by Whewell, is discussed in subsection 2.3. Marshall's extension, using offer curves, is considered in subsection 2.4. An important lesson is that Walras and Marshall produced the same formal model, leading to their emphasis on multiple equilibria and stability issues, but used different diagrammatic approaches that are directly and simply linked. Indeed, reference may be made to a Mill/Whewell/Walras/Marshall model. However, they stressed different aspects of the model, so that the initial appearance is very different and the origins are not obvious.

Section 3 turns to the utility maximising foundations of exchange, starting in subsection 3.1 with the pioneering contribution of Jevons. Subsection 3.2 then examines Walras's approach, paying particular attention to their differing attitudes to the same fundamental 'equations of exchange'. Both Jevons and Walras concentrated on price-taking solutions to these equations. But Jevons left the equations in terms of quantities exchanged, leaving the equilibrium price ratio to be determined by the resulting ratio of quantities exchanged. Walras instead introduced the price at an early stage and thereby showed the route by which the general equilibrium demand and supply curves that he had produced earlier (in extending Cournot's model) can be derived. Subsection 3.3 then briefly discusses Edgeworth's treatment of exchange, representing the high point in the development of formal exchange models.

Section 4 discusses two closely related contributions to the literature, by Laukhhardt (1993) and Wicksell (1954), dating from 1885 and 1893 respectively. These can in many ways be regarded as masterly expositions of the theory (despite their lack of familiarity with Edgeworth's major contribution), although Laukhhardt made a number of original extensions of his own: indeed, it can be argued that his book represents the first major treatise on welfare economics. These contributions warrant closer analysis in view of the fact that, like Walras, they were
not translated into English for many years. Furthermore, Launhardt's reputation was damaged by unfair criticisms by Wicksell, who nevertheless relied heavily on the former's work. Brief conclusions are in section 5.

2 Non-utility Approaches

It is useful to begin a discussion of non-utility approaches with Cournot's (1927) attempt to examine 'trade' in a single good, involving two regions. On this analysis, Edgeworth later commented, not without sympathy, that 'the lesson of caution in dealing with a subject and method so difficult is taught by no example more impressively than by that of Cournot. This superior intelligence ... seems not only to have slipped at several steps, but even to have taken a wholly wrong direction.' (1925, ii, p.47). Its importance lies in the fact that Cournot's model provided an influential starting point for the development of a general equilibrium approach. A major value of his work seems to have been the stimulus it provided to Walras and Marshall to attempt to improve the basic model.

The formal similarity of the basic models used by Walras and Marshall, who both stressed multiple equilibria and examined stability properties, was mentioned above. The 'substantially equivalent' nature of the two analyses was stressed by Hicks, who suggested that, 'One feels almost obliged to explain it by the intrinsic excellence of the path they followed. Yet in fact there is a clear historical reason for it, one decisive influence we know to have been felt by both. Each of them had read Cournot' (1934, p. 346). Hicks's statement must, however, be qualified by the recognition that while Walras explicitly extended Cournot, Marshall extended Mill's treatment which itself provided such an impressive use of the basic ingredients of an exchange model and was produced almost ten years before that of Cournot. It does indeed seem that 'intrinsic excellence' played a major role.

2.1 Cournot's Trade Model

2.1.1 The Basic Framework

Cournot's (1927) framework was one in which a single good is initially produced in two countries that are isolated from each other. When 'communication' between the markets occurs, the good is produced and exported by the country in which it
is initially cheaper, allowing for transport costs. The market demand and supply curves were taken as given, and the regions have a common currency. In isolation the equilibrium price of the good is \( p_a \) and \( p_b \) in markets \( A \) and \( B \) respectively, with demand functions \( F_a (p) \) and \( F_b (p) \), and supply functions \( \Omega_a (p) \) and \( \Omega_b (p) \). The prices are given by the intersecting partial equilibrium curves and are the solutions to:

\[
\Omega_a (p_a) = F_a (p_a) \tag{1}
\]

and

\[
\Omega_b (p_b) = F_b (p_b) \tag{2}
\]

If \( p_a < p_b \) and the difference exceeds the cost of transporting the good between the two markets, \( \varepsilon \), then the good is exported from \( A \) to \( B \). Cournot argued that trade equalises the price of the good in the two markets, except for the transport costs. If the new equilibrium price in market \( A \) is denoted \( p'_a \), Cournot (1927, p.119) stated that this is given as the solution to:

\[
\Omega_a (p'_a) + \Omega_b (p'_a + \varepsilon) = F_a (p'_a) + F_b (p'_a + \varepsilon) \tag{3}
\]

so that total supply is equal to total demand in both markets combined. Cournot wrote:

\[
p'_a = p_a + \delta \text{ and } p'_b = p_a + \omega \tag{4}
\]

so that \( \delta \) is the change in the price in market \( A \) and \( \omega \) is the pre-trade absolute difference between prices in the two markets. Trade takes place only if \( \omega > \varepsilon \). Substitute for \( p_a = p_b - \omega \) in the first of the expressions in (4) and add \( \varepsilon \) to get:

\[
p'_a + \varepsilon = p_b + \delta + \varepsilon - \omega \tag{5}
\]

Equation (3) can then be re-written as:

\[
\Omega_a (p_a + \delta) + \Omega_b (p_b + \delta + \varepsilon - \omega) = F_a (p_a + \delta) + F_b (p_b + \delta + \varepsilon - \omega) \tag{6}
\]

This expression can be simplified using Cournot’s method of ‘development and reduction’ which involves taking the Taylor series expansion of each function of the form \( F(p + \delta) \) and neglecting squares and higher powers of \( \delta \). Thus:

\[
F (p + \delta) = F (p) + \delta F' (p) \tag{7}
\]
Expanding each term in (6) in this way, and using (1) and (2), Cournot (1927, p.120) obtained:

$$\delta \{ \Omega'_a(p_a) - F'_a(p_a) \} = (\delta + \epsilon - \omega) \{ F'_b(p_b) - \Omega'_b(p_b) \}$$

(8)

Demand curves are assumed to slope downwards and supply curves to slope upwards, so the term in curly brackets on the left hand side of (8) is positive, while that on the right hand side is negative. Since $\delta > 0$, then $\delta + \epsilon - \omega < 0$ and $\delta < \epsilon$. Hence the increase in price in market A must be less than the difference between the initial price differential and the unit transport cost.

Cournot used this model to examine the gains from trade using the concepts of consumer and producer surplus, and investigated the conditions under which the total demand in the two markets combined would increase. He also considered the question of whether the value of output would increase. In examining import or export taxes, Cournot made an algebraic slip which led him to believe that the price may fall in the importing country, although in fact price must rise in the importing, and fall in the exporting, country. This was briefly discussed by Edgeworth (1894, reprinted in 1925, ii, p.49), where he noted that Berry and Sanger, two former pupils of Marshall, had independently made the correction.

2.1.2 A Diagrammatic Version

Marshall (1975, ii, pp.246-248) made an early attempt to cast Cournot's model into diagrammatic form, mainly for the purpose of examining the gains from trade using measures of producers' and consumers' surplus. The diagrammatic analysis of the model was later refined by Marshall's student Cunynghame (1892, 1903). After an unsatisfactory start (1892, p.44), Cunynghame produced a 'back-to-back' diagram without any reference to Cournot but virtually paraphrasing the latter's introduction to his model (1903, p.317).

Ignoring transport costs, the diagram is shown in Figure 1 where the equilibrium price is such that $CT = EF$. Marshall's notes show the influence of Cournot on Marshall's analysis of consumers' and producers' surplus. Marshall's diagrams translate Cournot's surplus analysis into the now familiar triangles. Using the back-to-back version of Figure 1 the left hand side shows that the gains to $B$'s consumers arising from the price reduction outweighs the loss to producers, so that the net gain is equal to the shaded area $P_1CT$. The price increase in $A$
produces a net gain equal to the shaded area EPF in the right hand side of the figure. Marshall added that if in each country the cost of production is independent of output, then the exporting country gains nothing from trade (1975, ii, pp.247-8).\footnote{12}

2.2 Walras’s Extension

A fundamental criticism of Cournot’s model is that it deals with only one good and the analysis ignores the fact that money is used to purchase that good. This point was acknowledged by Cournot towards the end of the Researches, where he wrote that, 'It will be said that it is impossible for exportation of a commodity to fail to involve importation on the exporting market of a precisely equal value; and reciprocally, importation on a market involves exportation of an equal value ... It would be necessary to consider each of these nations as acting simultaneously the part of an importing nation and that an exporting nation, which would greatly complicate the question and lead to a complex result' (1927, pp.161-162).

It was left to Walras to make the extension.\footnote{13} Walras’s autobiography states that he ‘soon perceived’ that Cournot’s approach could not be applied to exchange and, ‘restricting my attention, therefore, to the case of two commodities, I rationally derived from the demand curve of each commodity the supply curve of the other and demonstrated how current equilibrium results from the intersection
of the supply and demand curves’ (Quoted in Jaffé, 1983, p.25). Walras’s transformation of Cournet’s model, using the same notation, is contained in (1954, pp.81-114). The crucial ingredient is the recognition that ‘to say that a quantity $D_a$ of (A) is demanded at the price $p_a$ is, ipso facto, the same thing as saying that a quantity $O_b$ of (B), equal to $D_a p_a$, is being offered’ (1954, p.88).\footnote{14}

Suppose that there are two goods, $X$ and $Y$, and comparative advantage is such that country $A$ exports good $X$ to country $B$, while the latter exports good $Y$ to $A$. Assume complete specialisation, and denote the relative price of good $X$ as $p$. This relative price can be interpreted as the amount of good $Y$ that must be given in order to obtain a unit of good $X$. For present purposes it is necessary to express $B$’s demand for $X$ and $A$’s demand for $Y$ as $F_b(p)$ and $F_a(p^{-1})$ respectively; $p^{-1}$ is the relative price of $Y$. The essential feature of an exchange model is that the demand for one good, at a given price, automatically carries with it a supply of the other good. $B$’s supply of $Y$, corresponding to the demand $F_b(p)$, is thus given by:

$$\Omega_b(p) = p F_b(p)$$

while $A$’s supply of $X$ is given by:

$$\Omega_a(p) = p^{-1} F_a(p^{-1})$$

The equilibrium price is that value of $p$ for which the demand for and supply of, say $Y$, are equal. This requires:

$$\Omega_b(p) = p F_b(p) = F_a(p^{-1})$$

which is equivalent to the equilibrium condition for good $X$, given by:

$$\Omega_a(p) = p^{-1} F_a(p^{-1}) = F_b(p)$$

The general equilibrium model therefore requires only the specification of the two demand functions in terms of the relative price, $p$; the associated supply curves are obtained using the reciprocal demand relationship. It was this insight that later led Wicksteed (1933) to argue that the concept of the partial equilibrium supply curve is ‘profoundly misleading’ and should be abandoned altogether.\footnote{15}
2.2.1 Linear Demands

In order to explore the nature of the model, suppose that demand functions are linear, such that:

$$F_b(p) = a - bp$$  \hspace{1cm} (13)

and

$$F_a(p^{-1}) = \alpha - \beta p^{-1}$$  \hspace{1cm} (14)

From A's demand for Y in (14), the corresponding supply of X is obtained using (31) as:

$$\Omega_a(p) = \alpha p^{-1} - \beta p^{-2}$$  \hspace{1cm} (15)

and equilibrium price is that which equates (15) and (13), giving:

$$\beta - \alpha p + \alpha p^2 - \beta p^3 = 0$$  \hspace{1cm} (16)

so that three equilibria, not necessarily real or distinct, exist. This approach therefore rapidly gives rise to the need to consider the stability of alternative equilibrium positions. The comparative static properties of models with multiple equilibria are of much interest, since small changes in demand conditions can lead to a large jump in the equilibrium price. The supply curve of X is 'backward bending' (if p is on the vertical axis), with supply reaching a maximum when \( p = \frac{2\beta}{\alpha} \) and a point of inflection where \( p = \frac{3\beta}{\alpha} \). Furthermore the maximum supply, where the price elasticity of supply is zero, occurs at a price for which the elasticity of demand (for Y) is minus one. It is the backward bending property that gives rise to the possibility of three equilibria.\(^{16}\) The diagrammatic representation of this model, using a simple modification of Figure 1, is shown in Figure 2.

The analysis may be extended by using (13) to write \( p = \{a - F_b(p)\}/b \). Substituting this expression for p into equation (30) gives:

$$\Omega_b(p) = F_b(p) \{a - F_b(p)\} /b$$  \hspace{1cm} (17)

Equation (17) has a simple interpretation as the 'offer curve' of country B, the concept introduced by Marshall. This offer curve is quadratic, so that if both countries have linear demand curves, the offer curves may intersect three times, consistent with the result from (16). It is well-known that the turning point of an offer curve occurs at the point of unit demand elasticity.
2.3 The Mill-Whewell Model

The previous subsection considered the path taken from Cournot to Walras. However, the same exchange model has a quite separate line of development, running from J. S. Mill and Whewell to Marshall. The concept of reciprocal demand combined with the clear idea of demand as a schedule was in fact explored by Mill almost a decade before Cournot’s book was published, although Mill did not publish his analysis of international trade until 1844. A mathematical analysis of Mill’s model was produced by Whewell in 1850. In his first published paper, Marshall (1876) indicated his preference for the general approach of Mill rather than Cournot, and his offer curves were directly stimulated by Mill’s analysis. Ironically, the precise nature of Marshall’s offer curves was misunderstood by Cunynghame, who went so far as to criticise Marshall’s analysis on the grounds that it should deal explicitly with more demand and supply curves (1903, p.317). Marshall commented rather tersely in a letter to Cunynghame that, ‘as to international trade curves – mine were set to a definite tune, that called by Mill’ (in Pigou, 1925, p.451).

2.3.1 Mill’s ‘Great Chapter’

In considering the determination of the terms of trade, between the comparative cost ratios of two countries, Mill was able to indicate the importance of recipro-
cal demand much more clearly than previous writers because of his conception of demand as a schedule. But Mill did not use mathematical notation, preferring to give numerical examples. The crucial element of the analysis is the idea that demand depends on relative prices. Hence England, assumed to have a comparative advantage over Germany in the production of cloth (while Germany has a comparative advantage in linen production), has a demand for linen that depends on its price relative to that of cloth. This relative price can be expressed in terms of an amount of cloth per unit of linen. This is the basis of Mill's argument that 'all trade is in reality barter, money being a mere instrument for exchanging things' (1920, p.583).

If England demands a certain quantity of linen, there is an associated, or reciprocal, supply of cloth equal to the amount of linen multiplied by the relative price. The quantity of linen multiplied by the amount of cloth per unit of linen obviously gives an amount of cloth. Neglecting transport costs, equilibrium requires that the post-trade relative price is the same in each country and that the price is such that Germany's import demand for cloth precisely matches England's export supply (associated with its demand for linen at the corresponding relative price). After giving a numerical example, Mill added that, 'as the inclinations and circumstances of consumers cannot be reduced to any rule, so neither can the proportions in which the two commodities will be interchanged' (1920, p.587).

He then went on, when considering the gains from trade, to add that 'the circumstances on which the proportionate share of each country more remotely depends, admit only of a very general indication' (1920, p.587). It is in Mill's subsequent discussion of such 'general indications' that he used the concept of demand elasticity – although he described it with the term 'susceptibility'. Mill effectively argued (1920, p.587-8) that if the German demand for cloth is completely inelastic, then all the gains from trade go to Germany. In general, Mill was able to demonstrate that, 'If, therefore, it be asked what country draws to itself the greatest share of the advantage of any trade it carries on, the answer is, the country for whose production there is in other countries the greatest demand, and a demand the most susceptible of increase from additional cheapness' (1920, p.591).
2.3.2 Whewell’s Mathematical Model

Whewell (1850) produced a mathematical version of Mill’s analysis, suggesting that the use of numerical examples led to the neglect of several important factors. Whewell denoted the English demand for linen and its relative price before trade as \( q \) and \( p \) respectively, with the German demand for cloth and its pre-trade relative price as \( Q \) and \( P \) respectively. After-trade prices and quantities are distinguished by the use of a dash, so that the equality of prices means that:

\[ p'P' = 1 \]

since, for example, \( P' \) is the reciprocal of the price of linen relative to that of cloth in Germany. The reciprocal supply of cloth implied by England’s demand for linen is, from the earlier argument, simply \( p'q' \). Hence trade balance is achieved when:

\[ p'q' = Q' \]

As a result of trade, the relative price of linen in England and that of cloth in Germany would fall by proportions \( x \) and \( X \) respectively, so that:

\[ p' = p(1 - x) \quad \text{and} \quad P' = P(1 - X) \]

If, in addition, the relative price of linen in Germany before trade is a proportion, \( k \), lower than in England, then \( 1/P = p(1 - k) \) and combining this with (20) and (18) gives:

\[ X = \frac{k - x}{1 - x} \]

A feature of Whewell’s model is his treatment of demand. He supposed that, for the English linen demand, the proportionate fall in the relative price, \( x \), would lead to a proportionate fall in ‘revenue’ (which in this context is the reciprocal supply of cloth) of \( mx \). The coefficient \( m \) was referred to as the ‘specific rate of change’ of the commodity, which Whewell supposed would vary over the demand curve. In fact it is possible to show that \( m \) is simply one plus the elasticity of demand.\(^{20}\) Similarly \( M \) is the ‘specific rate of change’ of the German demand for cloth. Hence:

\[ p'q' = pq(1 - mx) \quad \text{and} \quad P'Q' = PQ(1 - MX) \]
Substituting (21) and (22) into (19) gives, after rearranging, Whewell's result that:

\[ x = \frac{n(1 - Mk) - 1}{n(1 - M) - m} \]  

(23)

where \( n = \frac{PQ}{q} \) and is the value of cloth demanded in Germany before trade divided by the value of linen consumed in England before trade, measured in German prices. Alternatively \( n \) can be interpreted as the German pre-trade opportunity cost of cloth in terms of linen foregone, divided by the English demand for linen before trade. Whewell argued that when \( x = 0 \), England gains all the advantage from trade, and when \( x = k \) (that is, when \( X = 0 \)), Germany obtains all the advantage. It is therefore possible to obtain the limiting values of \( n \) under which trade takes place. Whewell's result gives a precise expression from which the terms of trade can be obtained in terms of the two demand elasticities and the relative 'sizes' of the two countries as measured by their demand. He also indicated that a country which receives no advantage from trade may well not specialise; that is, part of its relatively large demand for the imported good will be met from domestic production.

Whewell therefore made a significant advance over Mill's analysis. The main limitation of his approach is the highly restrictive specification of demand, which is strictly limited to small price changes. It is likely that Whewell's results provided the major impetus for Mill's supplementary sections, added in a later edition. Although Mill (1920, p.586) only mentioned Thornton, he had other 'intelligent criticism' in mind. Thornton may well have argued that the type of 'equilibrium' indicated by Mill might not be unique, a point Mill acknowledged at the beginning of the supplementary sections.\(^{21}\) This might have been reinforced by the recognition, indicated by Whewell, that \( m \) and \( M \) would generally vary along the respective demand curves. Mill's attempt to deal with non-uniqueness is of interest. However, there is much stronger contextual evidence that Mill's discussion of the limits within which his 'equation' applied, his recognition of partial specialisation and emphasis on the domestic transfer of resources came directly from Whewell; see in particular Mill (1920, p.598). Mill's discussion of inelastic, unitary, and elastic demands can easily be generated by appropriate substitution into (23) above.\(^{22}\)

In its essential components, Whewell's mathematical model of Mill is precisely the same as Walras's later generalisation of Cournot's one-good model. He spec-
ified only the two demand curves required, and used the concept of reciprocal demand and supply to generate the supply curves. The translation of Whewell’s model into diagrammatic form gives exactly the same as Figure 2 above. The major difference is that the detail of Whewell’s model was more cumbersome, because of his awkward specification of the functional relationship between price and quantity changes. It meant that he arrived at an early formal statement of the concept of price elasticity to which he made good use, but the simple specification used by Walras, modified from Cournot, was much more powerful.

2.4 From Mill to Marshall

It can be shown that a diagrammatic form of Whewell’s model leads directly to that used by Walras, from which it is a small step to derive the offer curves devised by Marshall. Marshall did not provide any statement of how he arrived at these curves, but it is of interest to compare Marshall’s attempt in (1975, i, pp.260-280) with the later analysis (1975, ii, pp.117-181), and finally with the mature version (1923, pp.330-360). He began in each case with basic offer curves for each country (that are elastic over the whole length shown) and then discussed their possible shapes in terms of elasticities. In the earliest essay, Marshall used the rather clumsy expression ‘guidance by the rate’ for ‘elasticity’. But by 1923 the analysis was clearly stated in terms of elasticities, and included a footnote giving the now standard geometrical method of finding the elasticity (1923, p.337, n.1). He also provided a method of deriving consumers’ surplus from the offer curves.

Marshall’s recognition of the possibility of his offer curves intersecting more than once, and the circumstances under which such multiple equilibria can occur, led him to devote much energy to dynamic adjustment problems and the question of which of several equilibria would be stable. Instead of presenting the mathematics of differential equations, Marshall applied, for the first time in economics, the now standard phase-diagram method. As usual, and after what must have been a great deal of thought on the question, Marshall was very sceptical about the use of mathematics to examine dynamic problems. Even if the equations of the offer curves were known precisely, he argued that, ‘the methods of mathematical analysis will not be able to afford any considerable assistance in the task of determining the motion of the exchange-index. For a large amount of
additional work will have to be done before we can obtain approximate laws for
representing the magnitude of the horizontal and vertical forces which will act
upon the exchange-index in any position' (1975, ii, p.163).28

Marshall came to regard his offer curve apparatus as capable of 'being trans-
lated into terms of any sort of bargains between two bodies, neither of whom is
subject to any external competition in regard to those particular bargains' (1923,
p.351). A major context was that of bargaining between firms and trade unions,
but it was left to Edgeworth to extend the analysis to those other areas.29 A
distinguishing feature of Marshall's analysis of offer curves was that he also had
variations in production in mind, rather than an exchange of fixed stocks; such
variations were clarified by the later detailed treatment of Meade (1952).

Marshall can also be seen at an early stage struggling with the problem of
'triangular barter'. In some 'pages from a mathematical notebook' (1975, ii,
pp.272-274), Marshall used demand curves specified in terms of relative prices
(similar to those considered above) to examine the situation in which Germany
exchanges linen for cloth, England exchanges cloth for fur, while Russia exchanges
fur for linen. Marshall's problem was very similar to the three-country case
considered by Mill, and the approach can be seen to follow Mill quite closely.
As Mill suggested, 'everything will take place precisely as if the third country
had bought German produce with her own goods, and offered that produce to
England in exchange for hers' (1920, p.592).30

2.4.1 Marshall and Whewell

The question arises of whether Marshall was directly influenced by Whewell; this
was first raised by Hutchison (1953, p.65). There is no reference to Whewell in
any of Marshall's writings on international trade; his only reference to Whewell
seems to be to the latter's role as editor of Richard Jones's works (see Pigou, 1925,
p.296 and 1975, ii, p.264). However, some writers have suggested that Marshall
made use of Whewell's work; these include Henderson (1985, p.422) and Cochrane
(1975, p.398). One argument to support this claim is that Marshall's signature
has been found on other volumes of the Transactions in which Whewell's papers
first appeared; see Collard (1968, p.xviii). Further references by Marshall to
Whewell have been collected by Vázquez (1995, p.249-250). But it does not seem
possible to attribute any particular analytical contributions of Marshall to the
work of Whewell.\textsuperscript{31}

While it may seem surprising that Marshall was not influenced with Whewell’s work, it is worth recalling a query raised by Hutchison (1950) in connection with Cournot’s Recherches. The possible significance of Cournot’s book was suggested to Jevons in 1875 by Todhunter who added that ‘I never found any person who had read the book’ (Hutchison, 1950, p.8). Yet Todhunter was, like Marshall, a Fellow of St. John’s College, and Marshall stated that he read Cournot in 1868. The lack of communication between Todhunter and Marshall on the subject of Cournot must have extended to Whewell, about whom Todhunter also had considerable knowledge.\textsuperscript{32}

3 Utility Approaches

3.1 Jevons’s Equations of Exchange

Jevons, in all respects a pioneer, presented his basic exchange analysis in the context of two traders, where $A$ and $B$ hold endowments, $a$ and $b$ respectively, of goods $X$ and $Y$. Where $x$ and $y$ are the amounts exchanged, utility after trade takes place can therefore be written as:

$$U_A = U_A(a - x, y)$$  \hspace{1cm} (24)

for trader $A$, while for $B$ it is:

$$U_B = U_B(x, b - y)$$  \hspace{1cm} (25)

Jevons actually used additive utility functions. The ‘keystone’ of the theory was the result that for utility maximisation, ‘the ratio of exchange of any two commodities will be the reciprocal of the ratio of the final degrees of utility of the quantities of commodity available for consumption after the exchange is complete’ (1957, p.95). This gives rise to his two famous ‘equations of exchange’, given using modern notation by:\textsuperscript{33}

$$-\frac{\partial U_A/\partial x}{\partial U_A/\partial y} = \frac{dy}{dx} = -\frac{\partial U_B/\partial x}{\partial U_B/\partial y}$$  \hspace{1cm} (26)

The term $dy/dx$ is the ratio of exchange of the two commodities at the margin. Jevons recognised that the integration of these differential equations presents
formidable difficulties, and for this reason he restricted his attention to price-taking equilibria. He used the analogy of a lever to stress that the movement of a lever out of equilibrium also requires the difficult treatment of differential equations, but that if attention is restricted to the properties of an equilibrium, 'no such integration is applicable' (1957, p.105).

The price-taking equilibrium was examined by using his 'law of indifference', such that there are no trades at disequilibrium ratios of exchange and 'the last increments in an act of exchange must be exchanged in the same ratio as the whole quantities exchanged' (1957, p.94). This means that \( y/x \) can be substituted for \( dy/dx \) in (25), giving the two simultaneous equations:

\[
\frac{\partial U_A/\partial x}{\partial U_A/\partial y} = \frac{y}{x} = -\frac{\partial U_B/\partial x}{\partial U_B/\partial y}
\]

(27)

Jevons recognised that \( y/x \) is equivalent to the ratio of prices of the two goods, \( p = p_x/p_y = y/x \), but he preferred to leave \( p \) out of the equations until the equilibrium values of \( y \) and \( x \) have been obtained. He recognised that in general the equations in (27) would be nonlinear and so not capable of explicit solutions. He therefore did not take their formal analysis further, although added the important but rather cryptic comment that the theory is 'perfectly consistent with the laws of supply and demand; and if we had the functions of utility determined, it would be possible to throw them into a form clearly expressing the equivalence of supply and demand' (1957, p.101). He went on to discuss a number of 'complex cases', involving large and small traders, three goods and three traders, and competition between two traders, showing a very confident handling of the use of the equations of exchange.\(^{34}\)

The discussion of price-taking behaviour (through the law of indifference) in the context of a two-person exchange model can be seen to create some 'tension' in view of the argument that there is no reason why two isolated traders should take prices as being outside their control. This point was raised by Jenkin before the publication of the *Theory of Political Economy* (see Black, 1977, iii, pp.166-178).\(^{35}\) It may have been in response to this criticism that Jevons introduced the 'trading body', defined as 'any body either of buyers or sellers' (1957, p.88), as a rather awkward device to concentrate on representative traders who are price-takers. Edgeworth (1881, p.109) later described the idea more clearly as 'a sort of typical couple'.
3.2 Walras, Utility and Demand

Walras's extension of the Cournot model in a non-utility framework has already been discussed in section 2. As he later stated, he 'proceeded to derive the demand curve itself from the quantities possessed by each individual in the market and from each individual's utility curves for the two commodities considered' (quoted by Jaffé, 1983, p.25). Hence Walras explicitly considered the step to which Jevons had alluded, but it is important to recognised that the demand and supply curves are not partial equilibrium concepts; they refer to general equilibrium curves such as those shown in Figure 2.

What is surprising is that his approach, and associated demand and supply curves, seem to have been almost entirely 'lost'; they do not appear in any history of economics or microeconomics texts. They received their most extensive development by Launhardt (1993), whose analysis was used heavily by Wicksell (1954), and is discussed in section 4 below.35

3.2.1 Demand and Supply Curves

Walras was able to make the link from utility to demand following the crucial advice of his colleague Paul Piccard; see Jaffé (1983, pp.303-305). What Piccard gave Walras was essentially the 'equations of exchange' that Jevons had earlier produced.37 The starting point is thus each equation in (27), and for trader A, the holder of good X:

\[
p = -\frac{\partial U_A}{\partial x} / \frac{\partial U_A}{\partial y}
\]  

(28)

Walras made the crucial step of recognising that if the substitution \( y = px \) is made where \( y \) appears anywhere on the right hand side of (28), it becomes an equation containing only \( p \) and \( x \). Hence it may be possible to solve for \( x \) as a function of \( p \), thereby giving \( A \)'s supply curve of good \( X \). Walras appeared to overlook the difficulty of solving the equation in practice and he did not examine any particular utility functions.

It is precisely at this point where Walras departed from Jevons, who preferred to leave the determination of the price ratio until the final stage, after obtaining the equilibrium amounts of goods \( X \) and \( Y \) traded. This created a problem because, as mentioned above, his equations are nearly always nonlinear and he fully recognised the problem of getting explicit solutions; see especially Jevons
(1909, p.759). This aspect has been ignored in the literature concerned with Jevons’s ‘failure’ to derive demand curves from utility maximisation.

This approach of Walras gives A’s supply function for good \( X \). The demand for \( Y \) is obtained using the reciprocal demand relation that \( y = px \). To get B’s demand for good \( X \), it is necessary to take the result that \( p = -\partial U_B / \partial x / (\partial U_B / \partial y) \) and again substitute for \( y = px \) and solve for \( x \) as a function of \( p \).

Although it is not always possible to solve the ‘equations of exchange’ for \( x \) and \( y \), an advantage of Walras’s approach is that his general equilibrium supply and demand curves can sometimes be derived explicitly. This enables the structure of the exchange model to be examined in some detail and its essential properties explored. The following subsection provides an example using the special case of constant elasticity of substitution utility functions.

### 3.2.2 An Example: CES Utility Functions

Suppose \( A \) has the constant elasticity of substitution utility function:

\[
U_A = \left\{ \alpha_1 (a - x)^{-\rho_1} + (1 - \alpha_1) y^{-\rho_1} \right\}^{-1/\rho_1}
\]

where \( \rho > -1 \), and \( \sigma_1 = 1 / (1 + \rho_1) \) is the elasticity of substitution between the two goods. Differentiating with respect to \( x \) and \( y \) gives:

\[
\frac{\partial U_A}{\partial x} = \frac{-\alpha_1 (a - x)^{-(1 + \rho_1)} U_A}{\left\{ \alpha_1 (a - x)^{-\rho_1} + (1 - \alpha_1) y^{-\rho_1} \right\}^{1/\rho_1}}
\]

\[
\frac{\partial U_A}{\partial y} = \frac{(1 - \alpha_1) y^{-(1 + \rho_1)} U_A}{\left\{ \alpha_1 (a - x)^{-\rho_1} + (1 - \alpha_1) y^{-\rho_1} \right\}^{1/\rho_1}}
\]

Hence:

\[
\frac{\partial U_A/\partial x}{\partial U_A/\partial y} = -\left( \frac{\alpha_1}{1 - \alpha_1} \right) \left( \frac{a - x}{y} \right)^{-(1 + \rho_1)}
\]

Person \( A \)’s supply of good \( X \), as a function of the price ratio, \( p \), is obtained by setting \( (\partial U_A/\partial x)/(\partial U_A/\partial y) \) equal to \(-p\), substituting for \( y = px \) and solving for \( x \). After some manipulation it can be found that:

\[
x = \frac{a}{1 + k_A p^{1 - \sigma_1}}
\]
where:

\[ k_A = \left( \frac{\alpha_1}{1 - \alpha_1} \right)^{\sigma_1} \]  

For person B, utility is:

\[ U_B = \left\{ \alpha_2 x^{-\rho_2} + (1 - \alpha_2) (b - y)^{-\rho_2} \right\}^{-1/\rho_2} \]  

where \( \sigma_2 = 1/(1 + \rho_2) \) is B’s elasticity of substitution. Using a similar process, it can be found that B’s demand for \( X \) is given by:

\[ x = \frac{b/p}{1 + \left( \frac{1}{k_B} \right) \rho^{\sigma_2 - 1}} \]  

where:

\[ k_B = \left( \frac{\alpha_2}{1 - \alpha_2} \right)^{\sigma_2} \]  

It is not possible to solve analytically for the equilibrium price for which A’s supply is equal to B’s demand for good \( X \). Inspection of the shape of the supply and demand functions shows, however, that there is only one equilibrium solution. The demand curve for \( X \) is always downward sloping and the supply curve is always upward sloping if the elasticity of substitution, \( \sigma_1 \), is greater than unity. If the elasticity is less than unity, then the supply curve is backward bending over the whole of the range, while the demand curve continues to slope downwards.\(^{39}\)

3.2.3 Walras Lost and Found

The subsequent neglect of Walras’s approach is unfortunate in view of its usefulness. For example, it shows how ‘backward bending’ supply curves can arise in a ‘natural way’; it was in the context of the backward bending supply curve of labour that Buchanan (1971) referred to the modern treatment in terms of ‘doctrinal retrogression’.\(^{40}\) The backward bending curves show immediately how multiple equilibria can arise. The recognition of such multiple equilibria led Walras to his famous analysis of stability.

The usefulness of the approach is perhaps also demonstrated by the fact that similar curves have been independently reinvented several times. For example, Viner (1955, pp.538-541) used similar curves to explain an international trade
argument of J.S. Mill (and in a long footnote derived the relationships between the relevant elasticities), although Viner did not show any backward bending curves. Vickrey (1964, pp.105-108) derived such curves directly from the Edgeworth box diagram. Atkinson and Stiglitz (1980, p.189) derived the general equilibrium curves directly from special utility functions suggested by Shapley and Shubik (1977); but they made no reference to Walras.

These ‘rediscoveries’ are of course in the context of exchange and general equilibrium. While modern general equilibrium theory has established the full conditions required for the existence and uniqueness of equilibrium, the development of the theory did not actually proceed in a direct line from Walras. Indeed, the early stimulus came from Cassel, who quickly arranged for his work to be translated into English and did not acknowledge that his simplified general equilibrium model was taken from Walras.41

3.3 Edgeworth: The Apogee

Edgeworth, directly stimulated by his personal contact with Jevons, provided a majestic synthesis and extension of the exchange model in his highly original Mathematical Psychics (1881). Jevons (1957, p.96) commented that ‘it is hardly possible to represent this theory completely by means of a diagram’, but of course Edgeworth provided such an apparatus with his indifference curves and contract curve contained within his ‘box diagram’.42 He also linked the offer curves directly to indifference curves. The technical device of the box diagram, after a very slow start, has now become ubiquitous in microeconomic theory.

A price-taking, or competitive, equilibrium is shown in Figure 3, which shows pre-trade indifference curves, offer curves, and the mutual tangency of indifference curves with the price line. Edgeworth emphasised the role of the number of traders, stressed that indeterminacy arises with small numbers so that there is a need for arbitration, and showed that the utilitarian objective, as a principle of arbitration, specifies a position on the contract curve and is acceptable to risk averse traders, thereby foreshadowing the later ‘neo-contractarian’ utilitarian approach.43 He showed how increasing the number of traders using barter, with individuals following a recontracting process in which provisional bargains can be broken and coalitions can be formed, causes the range of indeterminacy along the contract curve to shrink until, with many traders, only the price-taking equilibria
remain. All this was achieved at great speed and expressed in a highly individual style; for further discussion, see Creedy (1986a)

4 Later Expositions

Two of the most comprehensive expositions of the theory of exchange were made towards the end of the century, by Launhardt (1993) and Wicksell (1954). Unfortunately it was many years before their works were translated into English. The first of these was Wicksell who, despite his strong criticisms of Launhardt, made extensive use of his book and cannot properly be understood without reference to Launhardt’s analysis. Walrasian supply and demand curves were first derived formally from utility functions by Launhardt (1993) in 1885, who used the assumption of quadratic utility functions and demonstrated the properties diagrammatically much more clearly than Walras. He also used the results to examine disequilibrium trading and its welfare effects. Indeed Launhardt’s study can claim to be the first systematic treatise on modern welfare economics.

4.1 Launhardt’s Exchange Analysis

Launhardt’s analysis, starting from the exchange models of Jevons and Walras, is noteworthy for his derivation from explicit utility functions of algebraic forms of
general equilibrium supply and demand curves expressed as functions of relative prices. Whereas Jevons and Walras concentrated on the price-taking equilibrium properties of their exchange models, Launhardt explored a process of disequilibrium trading in which successive trades take place at the 'short end' of the market, that is, the minimum of supply and demand at a price. His main concern was, however, to examine the welfare aspects of exchange, comparing the gains from trade under competitive and monopolistic behaviour. Launhardt has been criticised for suggesting that aggregate utility, and thus the aggregate gain from trade, is maximised at the price-taking equilibrium. Launhardt nevertheless showed that a process of disequilibrium trading, in which the price initially favours the relatively poorer individual, can improve the aggregate gains from trade compared with the price-taking equilibrium.

4.1.1 Utility, Demand and Supply Functions

Instead of restricting attention to general results, Launhardt wished to illustrate the nature of the exchange model in more detail using explicit utility functions. Following Jevons, he assumed additive utility functions. As an engineer, it would have been natural to start with utility as a polynomial function of amounts consumed. The argument that marginal utility decreases steadily as consumption increases, with reference to Jevons's example of water, leads automatically to the quadratic form. Write A's utility as:

$$U_A (a - x, y) = \alpha_A (a - x) - \beta_A (a - x)^2 + \gamma_A y - \delta_A y^2$$  \hspace{1cm} (38)

Substituting \( y = px \) in (38) gives A's utility in terms of \( x \) and \( p \):

$$U_A = \alpha_A (a - x) - \beta_A (a - x)^2 + \gamma_A px - \delta_A (px)^2$$  \hspace{1cm} (39)

A's supply function can be derived by maximising \( U_A \) with respect to \( x \), for given relative price, \( p \). Setting \( dU_A/dx = 0 \) gives, after rearrangement:

$$x = \frac{\gamma_A p - (\alpha_A - 2\alpha\beta_A)}{2(\beta_A + \delta_A p^2)}$$  \hspace{1cm} (40)

A's demand for good \( Y \) is obtained by substituting into \( y = px \). If \( B \) has the utility function:

$$U_B (x, b - y) = \alpha_B x - \beta_B x^2 + \gamma_B (b - y) - \delta_B (b - y)^2$$  \hspace{1cm} (41)
then $B$’s demand for $X$ may be obtained by substituting $y = px$ into (41) and maximising $U_B$ with respect to $x$, giving:

$$x = \frac{\alpha_B - p(\gamma_B - 2\delta_B b)}{2(\beta_B + \delta_B p^2)}$$

Equation (40) and (42) are the equivalent of Launhardt’s results in (1993, pp.36-38).

Equating (40) and (42) shows that the equilibrium relative price is the root or roots of a cubic equation, but Launhardt (1993, p.43) made the simplifying assumption that the two individuals have identical tastes, differing only in their pre-trade endowments of the goods. Setting $\alpha_A = \alpha_B = \alpha$, and so on, the denominators of both (40) and (42) become identical, so that the term in $p^2$ cancels and the equilibrium price is:

$$p = \frac{\alpha - \beta a}{\gamma - \delta b}$$

Hence the price depends only on the parameters of the utility functions and the total stocks of the goods available.\(^{47}\) This result has important implications for the following subsection.

### 4.1.2 Disequilibrium Trading

Whereas previous writers restricted attention to price-taking behaviour, Launhardt examined the implications of disequilibrium trading. This was in terms of ‘repeated exchange’ in which, starting from a disequilibrium price, there is gradual adjustment towards an equilibrium. With excess demand or supply, trade is assumed to take place at the ‘short end’ of the market, the minimum of supply and demand. At each stage of the price-adjustment process, there is a change in the allocation of goods between the two individuals. But in the case of identical individuals, such changes can have no effect on the final equilibrium price because, as shown by (43), this depends only on the parameters of the common utility functions and the total stocks.\(^{48}\)

Another example of a situation in which trading at disequilibrium, or false, prices does not affect the equilibrium price was later produced by Marshall, although the precise structure of Marshall’s example only became evident in the debate with Edgeworth.\(^ {49}\) If utility functions are additive and the marginal utility of one of the goods is constant, it can be shown that the final total amount
of the other good traded and the final relative price are independent of the sequence of disequilibrium trades. However, the amount traded of the good for which marginal utility is constant does depend on the sequence of trades. This result holds irrespective of the form of the individuals' utility functions for the good which does not have constant marginal utility.

The type of disequilibrium trading described by Launhardt, and later by Marshall, can be illustrated by an extension of a diagram suggested by Edgeworth; see Marshall (1961, p.844). This type of process was also discussed by Johnson (1913) who made no references to earlier literature. An example is shown in Figure 4, where the endowment position moves from $E$ to $E_1$, and then $E_2$. With the price line $EP$, there is an excess supply of good $X$ and trade takes place at the demand corresponding to point $E_1$. At the lower price, represented by the line $E_1P_1$ drawn through the new endowment point, the excess supply of $X$ is lower than formerly and the new trade takes place at the point $E_2$, the minimum of supply and demand. Each disequilibrium trade is a Pareto improvement and the sequence of trades, bounded by the pre-trade indifference curves, must converge to an equilibrium somewhere on the contract curve.
4.1.3 Individual Gains from Trade

Launhardt emphasised disequilibrium trading in order to examine its effect on the gains from trade. The assumption of common preferences made it much easier for him to provide numerical illustrations. His main focus was on the difference between alternative allocative mechanisms. He began by comparing the price ratio that maximises an individual’s gain from trade (with a single transaction taking place at that price) with the equilibrium price ratio. This is the equivalent of A’s ‘optimum tariff’ or monopoly price. Consider A, who begins by holding all the stocks of good X, and achieves a gain in utility resulting from trade, $G_A$, of:

$$G_A = U_A(a-x,y) - U_A(a,0)$$

$$= \gamma_{Ay} - \delta_{Ay}^2 - (\alpha_A - 2\beta_A a) x - \beta_A x^2$$

(44)

Substituting $y = px$ into (44) gives Launhardt’s (1993, p.46) result that:

$$G_A = x \{ \gamma_{Ap} - (\alpha_A - 2\beta_A a) \} - x^2 \{ \beta_A + \delta_{Ap}^2 \}$$

(45)

After producing the equivalent of (45), Launhardt substituted numerical values for the coefficients in utility functions and used the assumption of identical tastes in order to obtain $G_A$ in terms of $p$ and $p^2$. The value of $p$ which maximises $G_A$ turns out to be the positive root of a quadratic. However, he did not explain the precise method, giving only the numerical solution. Launhardt showed with his numerical examples that the resulting price ratio is different from the price-taking equilibrium value. However, the total gain of both A and B at that point is less than at the price-taking equilibrium.

Further insight can be obtained by differentiating (45) with respect to $p$, which gives, after collecting terms:

$$\frac{dG_A}{dp} = x \gamma_A - 2x^2 \delta_{Ap} + \frac{dx}{dp} \{ \gamma_{Ap} - (\alpha_A - 2\beta_A a) - 2x \{ \beta_A + \delta_{Ap}^2 \} \}$$

(46)

By taking the term $2(\beta_A - \delta_{Ap}^2)$ outside the curly brackets in (46) it can be seen that:

$$\frac{dG_A}{dp} = \gamma_A x - 2x^2 \delta_{Ap} + 2(\beta_A + \delta_{Ap}^2) \left\{ \frac{\gamma_{Ap} - (\alpha_A - 2\beta_A a)}{2(\beta_A + \delta_{Ap}^2)} - x \right\} \frac{dx}{dp}$$

(47)

The first term in the curly brackets in (47) is equal to A’s supply of good X at price $p$. However, it is important to recognise that the $x$’s in (47) must refer to
the demand for good $X$ by person $B$. This is because, for disequilibrium prices, trading must take place at the ‘short end’ of the market, and for prices that are favourable to person $A$, there is an excess supply of good $X$. By setting $dG_A/dp$ equal to zero, and substituting for $x$ from $B$’s demand function in (42), the price that maximises $A$’s gain from trade is the root of a rather complex expression. It is, however, clear that the equilibrium price, for which the term in curly brackets in (47) is zero, does not correspond to the price for which $G_A$ is a maximum, for which the whole of the right hand side of (47) must be zero.

If individuals are identical except for their endowments of goods, the gains from trade are equal if all trade takes place at the price-taking equilibrium price. This can be seen by writing $B$’s gains as:

$$G_B = U(x, b - y) - U(0, b) = x \{\alpha_B - (\gamma_B - 2\delta_B b)p\} - x^2 \left(\beta_B + \delta_B p^2\right)$$  

(48)

Substituting for the price-taking equilibrium price from (43), it can be seen from (45) and (48) and remembering that the supply and demand for $X$ must now be equal, that $G_A = G_B$. Launhardt (1993, pp.45-46) did not give such a proof, but showed that the gains are equal using his numerical example and pointed out that the result holds only for the special assumptions.

4.1.4 Price-taking and Aggregate Utility

Launhardt’s examination of exchange contained the unfortunate and incorrect argument that, ‘It is simple to prove that in an exchange at equilibrium the sum of the utility for both proprietors, that is the utility achieved in an overall economic sense, has reached a maximum’ (1993, p.43). Launhardt’s discussion was terse, with a lack of clarity that is uncharacteristic of his work.50 But he immediately qualified this statement, and the qualification was based on his analysis of a sequence of trades rather than a single exchange transaction. Launhardt’s slip provided an easy target for later critics such as Wicksell.

Launhardt would have avoided the problem if he had attempted to solve for the values of $x$ and $y$ which maximise $U$. Total utility, as above, is given by:

$$U = U_A + U_B = U_A(a - x, y) + U_B(x, b - y)$$  

(49)

In view of the assumption of additivity, the partial derivatives $\partial(U_A + U_B)/\partial x$ and $\partial(U_A + U_B)/\partial y$ depend only on $x$ and $y$ respectively. Hence the two first-
Table 1: The Pre-trade Situation

<table>
<thead>
<tr>
<th></th>
<th>Person A</th>
<th>Person B</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock of good X</td>
<td>400</td>
<td>0</td>
<td>400</td>
</tr>
<tr>
<td>Stock of good Y</td>
<td>0</td>
<td>480</td>
<td>480</td>
</tr>
<tr>
<td>Utility</td>
<td>240</td>
<td>576</td>
<td>816</td>
</tr>
</tbody>
</table>

Order conditions for the maximisation of $U$ can be solved explicitly for $x$ and $y$, as:

$$x = \frac{(\alpha_B - \alpha_A) + 2\beta_A a}{2(\beta_A + \beta_B)}$$  \hspace{1cm} (50)

$$y = \frac{(\gamma_A - \gamma_B) + 2\delta_B b}{2(\delta_B + \delta_A)}$$  \hspace{1cm} (51)

These values of $x$ and $y$ do not equal the competitive equilibrium values. The competitive equilibrium involves voluntary trading on the part of individuals attempting to maximise their utility, subject to given prices, whereas alternative solutions involve the amounts traded being imposed on individuals.\(^{51}\)

Launhardt assumed that the values of $\alpha$, $\beta$, $\gamma$ and $\delta$, the parameters of common utility functions, are 1.0, 0.001, 1.8 and 0.00125 respectively.\(^{52}\) The pre-trade situation is shown in Table 1 where, for example, $A$’s pre-trade endowment of 400 units of good $X$ gives utility of 240.\(^{53}\) The price-taking equilibrium and the values which maximise aggregate utility, the utilitarian solution, are shown in the first two rows of 2. For identical tastes the utilitarian position involves equal sharing of the goods, and person $B$ is worse off than before any trade takes place. Despite the fact that aggregate utility is maximised, there is no constraint requiring non-negative gains. The equilibrium price for price-taking may be obtained from equation (43) as 0.5. The assumption of identical preferences means that both individuals obtain a gains from trade of 93.33. Aggregate utility is less than with the utilitarian arrangement, as shown in the final column of 2.\(^{54}\)

Launhardt contrasted the equilibrium with a sequence of trades and monopoly pricing. For the latter, substitution into $A$’s gain from trade, given by equation (45), using $B$’s demand for $X$ from equation (42) to substitute for $x$, gives rise to $A$’s gain in terms of the price ratio as:

$$G_A = \frac{-2520p^2 + 5040p - 1400}{4 + 5p^2}$$  \hspace{1cm} (52)
Table 2: Launhardt’s Sequence of Trades

<table>
<thead>
<tr>
<th></th>
<th>( p )</th>
<th>( x )</th>
<th>( y )</th>
<th>( U_A )</th>
<th>( U_B )</th>
<th>( U_A + U_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utilitarian</td>
<td>.5</td>
<td>200</td>
<td>240</td>
<td>520</td>
<td>520</td>
<td>1040</td>
</tr>
<tr>
<td>Price-taking</td>
<td>0.5</td>
<td>266.67</td>
<td>133.33</td>
<td>333.33</td>
<td>669.33</td>
<td>1002.66</td>
</tr>
<tr>
<td>Trade 1.</td>
<td>.78</td>
<td>151.09</td>
<td>117.85</td>
<td>381.73</td>
<td>616.19</td>
<td>997.92</td>
</tr>
<tr>
<td>Trade 2.</td>
<td>.5</td>
<td>95.43</td>
<td>47.71</td>
<td>393.68</td>
<td>628.14</td>
<td>1021.82</td>
</tr>
<tr>
<td>Trade 1.</td>
<td>.43</td>
<td>233.12</td>
<td>100.24</td>
<td>306.91</td>
<td>682.07</td>
<td>988.98</td>
</tr>
<tr>
<td>Trade 2.</td>
<td>.5</td>
<td>41.32</td>
<td>20.66</td>
<td>309.15</td>
<td>684.31</td>
<td>993.46</td>
</tr>
</tbody>
</table>

Launhardt (1993, p.46) gave this equation and stated without discussion that \( G_A \) reaches a maximum for \( p = 0.78 \).\footnote{55} The situation after trading at this price is shown in the third row of Table 2, where person \( A \) has utility of 381.73 which exceeds that obtained from the price-taking equilibrium. Trader \( A \) gains 141.73 while \( B \) gains 40.19 from the trade at \( p = 0.78 \). However, aggregate utility is less than at the price-taking equilibrium.

This single trade does not exhaust all the potential gains from trade, and further trades can take place at prices lower than 0.78. Launhardt assumed that the next trade took place at the equilibrium price, and with identical tastes this is independent of the earlier trade. Hence the second trade occurs at \( p = 0.5 \) and the result is given in the fourth line of Table 2. The aggregate gain as a result of the two trades is 1021.82 and is higher than in the price-taking. It is indeed this result which gives Launhardt’s qualification to his earlier argument. A similar approach shows that the monopoly price set by person \( B \) is \( p = 0.43 \).

The resulting sequence of trades is shown in the last two rows of Table 2. The sequence of the two trades is shown in Figure 5.\footnote{56}

### 4.2 Wicksell’s Examples

Wicksell (1954) relied heavily on Launhardt’s treatment. However, he saw clearly that aggregate utility is not maximised at the price-taking equilibrium, and for this reason was strongly critical of Launhardt; see Wicksell (1954, p.18). However, Wicksell’s discussion of disequilibrium trading can only be understood with reference to Launhardt’s treatment. Wicksell produced his own numerical examples of a sequence of trades, assumed values for \( \alpha, \beta, \gamma \) and \( \delta \) of 200, 5, 10 and 0.5 respectively, and supposed that trader \( A \) holds 10 units of \( X \) while \( B \) holds 100 units of \( Y \).\footnote{57}
Figure 5: Launhardt's Sequence of Two Trades

Substituting into the equivalent of (43), Wicksell obtained the equilibrium price ratio of 30. Substitution for \( p = 30 \) into equation (40), and using \( y = 30x \), gives equilibrium amounts traded as \( z = 2 \) and \( y = 60 \), where both traders gain 200 units of utility.\(^{58}\) Wicksell did not present the equations of the demand and supply curves, however. Instead of goods \( X \) and \( Y \), he used oxen and sheep respectively and stated, 'supposing that he \( |A| \) was first expected to exchange 1 ox for 13 sheep, then a second ox for \( 17 \frac{2}{3} \) sheep, then \( \frac{1}{2} \) ox for 11 sheep and finally another \( \frac{1}{2} \) ox for 15 sheep, then there would remain for him after each exchange respectively a proportion of exchange between sheep and oxen [marginal rate of substitution] of more than 1 : 13, 1 : \( 17 \frac{2}{3} \), 1 : 22 and finally of just 1 : 30, so that each single exchange would have to seem to him undoubtedly profitable, although he has in fact finally exchanged just 3 oxen for not quite 57 sheep' (1954, p. 67).

The final result is that \( B \) gets more oxen (good \( X \)) and gives up fewer sheep (good \( Y \)) in comparison with the price-taking equilibrium. Wicksell's terse discussion of this sequence of disequilibrium trades is misleading and conceals much numerical work. For example, he supposes that \( A \) is first expected to exchange 1 ox for 13 sheep. But if the price ratio is 13, it is shown in Table 3 that 1.115 units of \( X \) are traded for 14.498 units of \( Y \). The ratio, \( y/x \), is equal to the price ratio, but the trade that person \( A \) prefers is not quite the same as in Wicksell's example. Similarly, when the price is \( 17 \frac{2}{3} \), \( A \) does not actually wish to trade 1
Table 3: Wicksell's Example

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>x</th>
<th>y</th>
<th>$U_A$</th>
<th>$U_B$</th>
<th>$U_A + U_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utilitarian</td>
<td>-</td>
<td>5</td>
<td>50</td>
<td>1250</td>
<td>1250</td>
<td>2500</td>
</tr>
<tr>
<td>Price-taking</td>
<td>30</td>
<td>2</td>
<td>60</td>
<td>1700</td>
<td>700</td>
<td>2400</td>
</tr>
<tr>
<td>Trade 1.</td>
<td>13</td>
<td>1.115</td>
<td>14.498</td>
<td>1516.73</td>
<td>706.32</td>
<td>2223.05</td>
</tr>
<tr>
<td>Trade 2.</td>
<td>17 2/3</td>
<td>0.968</td>
<td>17.105</td>
<td>1536.05</td>
<td>845.05</td>
<td>2381.10</td>
</tr>
<tr>
<td>Trade 3.</td>
<td>22</td>
<td>0.508</td>
<td>11.165</td>
<td>1543.57</td>
<td>893.17</td>
<td>2436.74</td>
</tr>
<tr>
<td>Trade 4.</td>
<td>30</td>
<td>0.458</td>
<td>13.736</td>
<td>1554.05</td>
<td>903.65</td>
<td>2457.70</td>
</tr>
</tbody>
</table>

ox for $17 \frac{2}{3}$ sheep, but 0.968 ox for 17.105 sheep. There is also a discrepancy in the other trades, given the prices mentioned by Wicksell, and in total 3.049 ox are traded for 56.504 sheep. Following the sequence of disequilibrium trades, $A$ is slightly better off than in the pre-trade position, having gained 54.05 units of utility from the trades overall, but $B$'s gain from trade is, at 403.65, over double that obtained from the price-taking equilibrium. The conclusion is similar to that of Launhardt. When the disequilibrium trades favour the poorer person, in this case trader $B$, total utility is greater than in the competitive solution. The fact that Wicksell did not make it clear that he had followed Launhardt and uncharacteristically poured scorn on his work simultaneously made his own contribution less transparent and damaged the reputation of Launhardt.

5 Conclusions

The aim of this paper has been to provide an outline of the development of the theory of exchange, concentrating on the less well-known development of the formal model which culminated in the contribution of Edgeworth. The importance of exchange, viewed as the central economic problem for the early neoclassical economists, was stressed. Instead of taking a chronological approach, non-utility approaches were first discussed. These included the extension by Walras of Cournot's attempt to model trade between regions, and Whewell's mathematical version of J. S. Mill's international trade analysis, followed by Marshall's diagrammatic version. Jevons's and Walras's utility approaches were then examined, showing the different paths they took from the same basic equations of exchange. After a very brief discussion of Edgeworth, the neglected but valuable contribution of Launhardt, along with the later work of Wicksell, were examined.
Emphasis was placed on the similarity of the formal structure of the exchange model used by the various writers. This similarity has been obscured by the different forms of presentation used and the emphasis given to various aspects and results by each investigator.

The pioneers of the theory of exchange fully recognised that they were dealing with pure theory rather than the 'real world'. Despite their enthusiasm over the successful development of a theory which helped to clarify the nature of alternative equilibria and provided a vehicle for the analysis of the welfare implications of exchange, they were conscious of its limitations. Indeed, Edgeworth himself managed to combine the greatest enthusiasm for the abstract results with the greatest diffidence regarding their applicability. This led to a stress on 'negative' results, or the removal of fallacies, and he quoted more than once the lines 'reason is here no guide, but still a guard'.59 This advice was not always followed by those making further refinements to the model; but that is another story.
Notes

1Schumpeter (1955, p 911) wrote, 'they realised the central position of exchange value' which 'is but a special form of a universal coefficient of transformation on the derivation of which pivots the whole logic of economic phenomena'. In considering the central position of exchange theory, Fraser (1937, p.104) stated that the view of costs in terms of foregone alternatives is 'merely the extension of the exchange relationship to the whole of economic life'.

2As Edgeworth stressed, 'the fundamental principle of international trade is that general theory which Jevons called the Theory of Exchange ... which constitutes the "kernel" of most of the chief problems of economics' (1925, ii, p.6). He added, 'distribution is the species of exchange by which produce is divided between the parties who have contributed to it' (1925, ii, p.13).

3Hicks also stated that 'welfare economics was captured by the catallactists and it has never got quite free' (1964, p. 253).

4See also Jevons’s letter to his sister in Black (1977, ii, pp. 361, 410). Schumpeter argued that the utility analysis must be understood in the context of exchange as the central 'pivot', and 'the whole of the organism of pure economics thus finds itself unified in the light of a single principle - in a sense in which it never had before' (1954, p.913).

5The famous expression regarding adjective and noun is from Hutchison (1953).

6Hicks (1964, p.250) suggested that the term marginal revolution, 'is a bad term, for it misses the essence of what was involved'.

7It is recognised that some commentators would dispute this point, placing much stress on different interpretations of Walras's famous tâtonnement process. But in the formal models it is hard to escape the fact that, just as in Jevons's approach, individuals are price-takers and that in the equilibria considered, all exchange takes place at the corresponding prices; trading at disequilibrium prices is considered in section 4 below.

8However, some economists, including Jaffé (in Walras, 1954, p.504) have argued that the approaches were different.

9This can be shown to depend on the ratio of demand to supply elasticities in each country, though of course Cournot did not use the concept of elasticity himself. For further details, see Creedy (1999b).

10The error was later also pointed out by Fisher, in Cournot (1927, p.xxiv).

11He suggested that 'the method of treating economics graphically is probably due to Cournot' and added, 'the chief credit of reviving an interest in this method rests with Professor Marshall' (1892, p.36). This work culminated in Cunynghame's book (1904), reviewed at length by Edgeworth (1905). It does not seem to be widely recognised that Cunynghame's treatment stems directly from Cournot. Even Viner (1953, pp. 589), who referred to Barone's use of the same diagram to measure the gains from trade, did not seem to recognise that the diagram represents Cournot's model. The origins were, however, recognised by Samuelson (1952).

12Although these notes are not dated, there seems little doubt that Cournot was the sole influence and that Jenkin's (1871) analysis was quite independent, as Marshall himself always insisted.
13This cannot be achieved simply by adding another good and imposing a balance of payments constraint. Additional partial equilibrium demand and supply curves cannot by their very nature cope with the interdependence which is at the heart of the problem.

14Jaffé (1983, pp.55-77) argued that the line of filiation is instead from Isard to Walras. Isard recognised the important point that the price ratio is equivalent to the (inverse) ratio of quantities exchanged. He also addressed the mutual interdependence in a general equilibrium system. But his discussion was restricted merely to given quantities; there is no analysis of demand as a function of relative prices. A section from Isard's analysis is reproduced in Baumol and Goldfeld (1968, pp.255-257), who suggest that strong claims made for him are 'somewhat over-enthusiastic' (1968, p.253).

15Wicksteed emphasised the role of endowments, or stocks, but surprisingly he did not use the apparatus of Walras and Jevons with which he was so familiar; see Creedy (1991a).

16On the equilibrium properties of this model and its implications for the distribution of prices, see Creedy and Martin (1993, 1994).

17Walras did not appear to be influenced by Mill's trade model, despite his familiarity with Mill's work. His published correspondence suggests that he did not become aware of Whewell's model until after he made contact with Jevons; see Jaffé (1965, I, letters 328, 375).

18See, for example, (1920, p.888). Mill's conception of demand in terms of a schedule is stressed by Robbins (1958, p.242) when comparing his trade analysis with that of Torrens, and by O'Brien (1975, p.183). Despite the importance of Torrens's and Pennington's work, Viner (1955, p.447) stresses the pivotal role of Mill's analysis for subsequent work. Although Pennington refers to the strength of demand when examining the gains from trade, he suggests (1840, pp.36, 39, 40-1) that the exchange rate will fluctuate between extremes, rather than tend to some determinate value. The high quality of Joplin's work in this area is now clear from O'Brien (1993, pp.211-219).

19Mill also introduced additional countries, transport costs, and additional goods, as well as examining the effects of technological change and shifts in demand; for further discussion of Mill and his critics, see Creedy (1990a).

20For a detailed analysis of Whewell's model, see Creedy (1989).

21Thornton's emphasis on indeterminacy in barter was also noted by Jevons and Edgeworth; see Creedy (1988a, p.48).

22Chipman (1965) interpreted Mill as assuming constant unit elasticities of demand; this is strongly criticised by Appleyard and Ingram (1979). On stability analysis, see Amano (1958).

23This was in fact the path taken by the present writer, in trying to see the link in diagrammatic terms between Mill and Marshall, without realising that the same diagram (or half of it) was in Walras.

24Whewell's treatment led to a certain amount of difficulty for later commentators; see Creedy (1989).

25For a detailed discussion and explanation of the diagrammatic links between the models, see Creedy (1990a).
26Consider England's offer curve in which cloth and linen are on horizontal and vertical axes respectively. If $T$ and $M$ are respectively the points where the tangent to the offer curve and a vertical line dropped from the point of tangency cut the horizontal axis, then the elasticity is $OM/OT$. The two points obviously correspond when the elasticity is unity.

27For further analysis of this method, see Creedy (1991b).

28For Samuelson's treatment of dynamics, see (1948, pp.266-269). Later treatments include Bhagwati and Johnson (1960) and Amano (1968). The latter concentrates on stability conditions.

29Marshall (1975, ii, p.112) contains a letter to Edgeworth of March 1891 in which Marshall discusses the application to wage bargaining. Edgeworth's application came as early as 1881, and was directly stimulated by Marshalls and Marshall (1879) and the privately printed chapters from the Pure Theory; see Creedy (1986a). Marshall also mentioned such applications in his 1878 paper on Mill; see Pigou (1925, pp.132-133).

30For further analysis of this case, see Creedy (1990a).

31Whewell (1850) showed that the King/Davenant 'law of demand' follows a third-order polynomial, yet when Marshall discussed the 'law' in the Principles, he simply reproduced some of Jevons's arguments; for further detail see Creedy (1986b). Whitaker (1975, i, p.45, n.26) noted that Marshall made no reference to Whewell's criticisms of Ricardo, but see Vázquez (1995, p.249).

32See, for example, Todhunter (1876). Whewell's correspondence shows that he was aware of Cournot. On the awareness of Cournot's work, see Vázquez (1997).

33For Jevons's version, see (1957, p.100). Jevons did not make use of constrained optimisation methods, but if utility is maximised subject to a 'price-taking' constraint, written as $y = px$ (from $xp_y = xp_x$), the Lagrangian is $L = UA + \lambda(y - px)$. The partial derivate $\partial U_A/\partial x$ is not marginal utility, but its negative.

34For further discussion of these cases, see Creedy (1992a).

35Jevons's earlier treatment was in terms of trade between two individuals called Jones and Brown. Jenkin (1870) stimulated Jevons to publish his own work quickly, though his non-mathematical statement had been made in 1862; see (1857, appendix III).

36The curves were discussed very briefly, in the comprehensive review of Walras's equilibrium economics, by van Daal and Jolink (1993, p.26). They commented that 'it did not get much following', and referred to the 'undeniable complexity of the figures'. The only treatment in general works on the history of economic analysis seems to be the brief mention by Stigler (1965, p.96), who also referred to Wicksell (but not to Launhardt).

37Neither Jevons nor Walras made use of the Lagrangean method of constrained maximisation; see Creedy (1986).

38An alternative approach would involve substituting $y = px$ into the utility functions, so that for example $U_A = U_A(a - x, px)$. Differentiating with respect to $x$ and setting the result to zero also gives, after rearrangement, $A$'s supply of good $X$ as a function of $p$. This approach avoids the use of Lagrange multipliers.
A richer range of possibilities exists if individuals hold some of both goods before trade takes place. For example, if person A holds $a_1$ and $b_1$ respectively of goods $X$ and $Y$ before trade, then in the supply curve of good $X$ the numerator is changed from $a$ to $a_1 - b_1kAp^{-\sigma_1}$. If person B holds $a_2$ and $b_2$ of $X$ and $Y$ respectively, then it can be found that the demand curve has a numerator of $(b_2/p) - (a_2/k_B)p^{\sigma_2-1}$ instead of simply $b/p$. While the demand curve is still downward sloping for all values of $\sigma_2$, the supply curve has both upward sloping and backward bending sections, so long as $\sigma_1 < 1$. In this case it is possible for multiple equilibria to occur.

It is possible, though unlikely, that Robbins's (1930) treatment of the supply curve of labour was influenced by Walras, indirectly through Wicksell, with whose work he was much more familiar. The direct influence of Wicksteed (1933) on Robbins is most likely, but even here there may be an indirect influence of Walras.

See, for example, Weintraub (1985). Even Phelps Brown (1936) based his exposition of general equilibrium on Cassel, without being aware that it was from Walras.

The treatment by Bowley (1924) is well-known, but is even more terse than Edgeworth (1881).

For a comparison of alternative types of exchange equilibria, including price-taking, utilitarian and bargaining solutions, see Creedy (1994b).

Neither writer seems to have read Edgeworth (1881), although Wicksell was familiar with the contract curve from Marshall’s *Principles*.

See for example, Wicksell (1954, p.76, n.2; 1934, p.81, n.1). For further discussion of the utilitarian optimum, along with price-taking and bargaining solutions, see Creedy (1994b).

The marginal rate of substitution is a ratio of linear functions of $x$ and $y$, so it is known, following the later work of Allen and Bowley (1935), that the expenditure on each good is a linear function of total expenditure, with coefficients depending on prices.

Further insight into the price-taking equilibrium in this special case, not discussed by Launhardt, can be obtained by noting that $A$’s marginal utility of good $X$ is equal to $\alpha - 2\beta(\alpha - x)$, while $B$’s marginal utility is $\alpha - 2\beta x$. The arithmetic mean marginal utility is thus $\alpha - \beta a$, the numerator of (43). Similarly, the arithmetic mean utility of good $Y$ is the denominator of (43). Hence the equilibrium price is the ratio of the arithmetic mean marginal utility of good $X$ to that of good $Y$.

This result does not hold if the individuals have different tastes: see Creedy (1994a).

See Creedy (1990b).

For further analysis of Launhardt’s spurious argument, see Creedy (1994a).

When criticising Launhardt, Wicksell (1934, p.81, n.1) pointed out that, with identical tastes, total utility is maximised when the parties ‘simply exchange half their stocks’. This is confirmed by appropriate substitution into (50) and (51) giving a rate of exchange, $y/x$, of $b/a$, which obviously differs from that resulting from equation (43) above.

He stated these in terms of fractions, which would have been more convenient when making calculations with pencil and paper.
The values in the tables, obtained using a computer, show that Launhardt's own calculations were accurate, though they would have been tedious to produce.

The utilitarian arrangement was not examined directly by Launhardt since he had incorrectly concluded that it coincides with the price-taking equilibrium.

Differentiation of $G_A$ with respect to $p$ and rearrangement shows that this price can, as mentioned above, be obtained as the positive root of a quadratic.

The monopoly price involves a tangency position between an indifference curve of the monopolist and the other trader's offer curve, but this is not on the contract curve so the second trade involves a movement to the contract curve. Launhardt also examined the limits of the contract curve, the intersection of the linear contract curve with quadratic pre-trade indifference curves.

He stated that the equation of the contract curve is $10x + 3y = 200$.

There is a misprint in Wicksell (1954, p.66) which gives $x = 30y$.

The source, not of course given by Edgeworth, is Pope's *Essay on Man* (second Epistle, argument III).
References


Todhunter, I. (1876) *William Whewell. An Account of his Writings, with Selections from his Correspondence*. London; Macmillan.


