C. L. Dodgson, Public Choice, and The Mathematics of Proportional Representation

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Introduction

Proportional representation (PR) voting schemes have been debated in Great Britain, Australia, South Africa and the United States for more than a century. Histories of the proportional representation movements such as Hart (1992) commonly associate PR most strongly with Thomas Hare and the Proportional Representation Society (P.R.S.). However, C. L. Dodgson (Lewis Carroll), a staunch conservative and able mathematician, not only proposed a form of PR differing from that advocated by Hare and the P.R.S., but his sophisticated analysis was an early form of what is now known as game theory. Mathematical analysis of voting schemes evolved into public choice and social choice theory, and Dodgson's work on PR remains an important, though neglected, landmark in the field.

The Reform Bill of 1832 which extended franchise in England also stimulated thought on the results of the system of voting used. From the 1830s on, a number of English voting theorists sought to construct a voting system whereby the party composition of parliament would more closely approximate that of the enfranchised country. Debate was quickened by the second Reform Bill of 1867 and by the extended franchise and redistribution of districts in 1884. This paper examines the analysis of C. L. Dodgson and shows its relationship with the constructs of game theory. Dodgson developed a two-stage game in which political parties use maximin strategy in a two person zero sum game after the government chooses an electoral method to minimize the maximum number of voters left unrepresented. This model has gone largely unnoticed, perhaps because Dodgson was primarily interested in proportional representation.

Nineteenth century analysis of voting centered around the issues of proportional representation, so that Dodgson's analysis as well as Hare's lies at the roots of public choice. Yet while modern public choice theorists have analyzed various voting schemes by determining their properties under various voter preferences and William Riker has used game theory to discuss strategic voting and coalition formation[1] PR schemes have been relatively neglected. Mueller (1989) analyzes the scheme most frequently associated with Thomas Hare and includes a section on empirical outcomes of PR, but does not examine the nineteenth century literature. Similarly, Tullock's (1972) discussion of PR does not address the early literature. Hart (1992) indicates that interest in proportional representation has not been constant, but has mushroomed periodically. Debate is currently active in Great Britain and the United States.

Duncan Black (1970, 28) referred to Charles Lutwidge Dodgson's work on voting theory as "the most distinguished contribution that has been made to Political Science since the seventeenth century". Due to Black's (1958) clear explication of their pioneering achievements, Dodgson's 1873, 1874, and 1876 pamphlets on the concepts of vote cycling and "disequity" are widely recognized, appearing explicitly in Mueller (1989). Yet although Black (1967, 1969, 1970) discussed Dodgson's later letters and pamphlet on PR, and Grofman (1989) largely follows Black, these publications are scarcely known even to choice theorists.

They embody the use of a game-theoretic model before von Neumann and Morgenstern (1944), as Black (1969) noted[2]. Moreover, in the first stage of the game Dodgson modelled, he used PR criteria to derive the appropriate number of seats per district and votes per elector. Although 19th century PR theorists had discussed the appropriate number of votes per elector and initiated limited experiments with limiting votes to less than the number of seats to be filled, their arguments were relatively heuristic.

In letters to the *St. James Gazette*, Dodgson examined the version of the Hare scheme advocated by the P.R.S. and raised other issues about the voting process. Most of these concerns were embodied in his last known work on voting theory, *The Principles of Parliamentary Representation* (PPR) (1884g with a second edition in 1885). Black (1970) gave an excellent account of Dodgson's activities and diary entries as related to his theory of voting and its motivation[3].

Letters to the *St. James Gazette*

Dodgson took his letters quite seriously. He sent copies of a letter on "Purity of elections" (*St. James Gazette*, 4 May 1881, largely reprinted in Collingwood, 1899, 233-4) "to Gladstone, Lord Salisbury, etc.". In this letter, he suggested that the desire to be on the winning side was a normal human motivation. To prevent it from influencing general elections which took several days, districts with polls closing early should have their ballot boxes sealed to preclude any announcement until all districts had returned. Salisbury replied that others had suggested a one-day election for the same purpose-- Dodgson's plan had the advantage of not partially disenfranchising voters with votes in more than one district. In discussing the purity problem in PPR, Dodgson specifically argued against "pluralists" ("like Cerberus, three gentlemen at once") (Dodgson 1884f, 35-6). Parliament adopted sealed ballot boxes in 1917 (Dodgson 1979, 429).

Dodgson (1884a) used the grounds of Hare's disciples to advance an objection to the single transferable vote. The P.R.S. suggested a single transferable vote: list voting by electors, with second, then third, up to nth choices being transferred to those candidates as preferred candidates were declared elected. Dodgson instanced a constituency with 11,999 voters, two seats and three candidates. If the votes were as follows, 5000 for A, then B; 3000 for A, then C; 1400 for B; and 2599 for C, then using the P.R.S.'s transfer scheme, A and C would be returned. Since 6400 voters prefer B to C and only 5599 prefer C to B, an injustice would have been perpetrated. Two replies, by Arthur Cohen on the following day and W. C. Sidgwick on the next, claimed that Dodgson had merely strengthened the P.R.S.'s arguments by showing that the minority would be represented. Dodgson (1884b) specialized his example, supposing that there were 5000 votes A B; 3000 votes A C; 1400 votes B A; and 2599 votes C A. A and C would still be returned, but 6400 voters want to elect A and B while only 5599 prefer A and C. There was no reply to this letter.

Dodgson (1884c) suggested a more restrictive and somewhat confusing version of criteria for transfer which would eliminate this problem. He noted that this method might not serve to fill all of a constituency's seats. In this case, he stated blithely, another method would have to be used.

Dodgson (1884d) advanced another instance of the failure of the single transferable vote, less successfully. He posited a 39,999 elector constituency with three seats and four candidates, Liberals A, B, C, and D and Conservative Z and supposed that there were 21,840 votes A B D; 10,160 votes A C B; and 7,999 votes Z. In this case A, B, and Z would be

elected under single transferable vote, although there is "no shadow of doubt A B C should be returned", as the minority for Z is not a quota. The next day Sidgwick responded that he did not think to practical politicians that it will seem a great grievance that it should be possible for a minority of one-fifth to return one member out of three when the majority are so foolish as to get four out of three. Even with no parties posited, this example too would instance the failure of the single transferable vote to achieve its ends.

Axioms for a system of voting

Despite cavilling at the P.R.S., Dodgson believed in the principle of proportional representation, though he thought the single transferable vote unlikely to achieve it. Dodgson (1884e, 1884f) set forth the models he would use virtually unaltered in the *Principles of Parliamentary Representation* (1884g). Dodgson (1884e) offered axioms which he would rely upon in developing the voting system which he repeated with some amplification PPR. These were

(1) that the numbers of electors per M.P. should be nearly equal across constituencies,
(2) that the minority should be adequately represented,
(3) that wasted votes should be avoided,
(4) that the ballot should be simple to mark, and
(5) that ballots should be simple to tot up.

Axiom (3) depends on Dodgson's definition of representation, which seems a bit odd today. Black (1969, 207-8; 1970, 11) argued that it came directly from James Garth Marshall's *Minorities and Majorities: Their Relative Rights* and Walter Baily's *Proportional Representation in Large Constituencies*. The definition starts with the concept of the Droop quota (Droop, 1868)-- the minimum number of votes a candidate in a plurality rule election must have to be certain of winning. Where V votes have been cast to elect m candidates, the Droop quota is the least integer greater than [V/(m+1)]. In a race for three seats where 1000 votes have been cast, a candidate may be sure she has been returned when 251 votes for her have been counted. Dodgson's idea, common to other 19th century voting theorists, was that if a candidate received more votes than the Droop quota, the extra votes were "wasted" and those who had cast them were not represented. Borcherding (1969) felt that PPR was not of great value, largely because this concept of representation is not the contemporary one. However, if a voter knew her most favoured candidate would be elected without her ballot, she would rationally prefer to vote for her second favourite.

Axioms (2) and (3) led to a calculation in a two-party country of the percentage of voters unrepresented (votes wasted) under varied numbers of members returned when the number of votes electors have vary from one to the number of seats. This and the second model discussed below were the first explicit analyses of the optimal number of members per district. Dodgson assumed two political parties, Reds and Blues, with 6/11 of the House Red and 5/11 Blue after an election, with a uniform distribution of Red and Blue electors in each district. Under these suppositions, he gave what he thought were the number of electors nationwide represented by Red members, by Blue, and not at all, but which was the expectation of such representation (Dodgson, 1884g, 23-27: Black, 1969, 209, noted that this is an expectation).

Dodgson's two-stage game

Dodgson's model was a game in the first stage of which the government sets m and v, and in the second of which a two-party election occurs. Duncan Black (1967, 1969) has recognized the election is worked out as a two person, zero sum game (which he claimed, strangely, was "isomorphic" to the problem of personal choice). Dodgson's criterion of minimizing the expected percentage unrepresented functions as a minimax criterion, as Black observed, though Black did not note that it was a criterion Dodgson gave the government. Black has acutely noted two assumptions underlying this model, which he did not mention were implicit rather than explicit in Dodgson's work. These are that Reds and Blues know their supporters (ergo, their number) and can make them vote to party program. Otherwise, Reds might all vote for candidate A in a four member district, while B, C, and D are elected by Blues even if Reds could have filled all seats. Thus he assumed that agents, in this case party organizers, optimize.[4]

Dodgson drew the conclusion that the government should find multiple seats preferable to singleton districts and give voters one vote each. Duncan Black's brilliant analysis of Dodgson (1884g) neglected the fact that both partyless and two-party models yield the same results.

As Black (1969) stated, Dodgson's approach employed criteria equivalent to minimax. Dodgson analyzed a game between two parties which seek to maximize the number of seats they can fill in a district regardless of the other party's strategy. Strategy consists of a vector [number of candidates, distribution of votes]. Dodgson assumed that parties put up a number of candidates equal to the number of seats the party desires to fill, while distribution of votes should always be as even as possible, subject to an integer constraint, which is optimal. (Despite this constraint, a pure strategy equilibrium always exists. As Black (1969) remarked, a mixed strategy is not meaningful in the context of an election.)

Dodgson modelled the m-seat, v-vote constituency, and gave criteria which would indicate to a party which knows its support how many seats it should attempt to fill. Given the strategic behaviour of the parties in the subgame, Dodgson analyzed a game in which a benevolent authority determines m and v with the objective of minimizing the number of electors unrepresented. Dodgson's work used subgame perfect equilibrium in analyzing this problem and the one discussed below, with the sets of players changing between problems.

The remaining question, not stated directly by Dodgson, was whether the percentage of voters unrepresented could be reduced further than by instituting one vote, multiple elector districts. That is, could the game yield better results for the benevolent authority if its strategy could be expanded beyond choosing (m,v) for each district. Dodgson's axiom (3) was the principle that as few votes as possible be "wasted" by being cast for a candidate who was to receive more than the Droop quota out of votes cast. Dodgson suggested that all candidates with more than the Droop quota be declared elected directly by the returning-officer. If any seats remained to be filled, all candidates would be called together by the returning-officer. At this meeting, those with extra votes and those with votes insufficient to elect them would be permitted to assign them to candidates not yet elected (Dodgson, 1884f, 36-40). Dodgson thought that, under these circumstances, all those whose votes (whether transferred or not) were used to seat a member would be represented. Dodgson suggested this to keep each ballot simple and because most voters probably had very incomplete preference lists, knowing only their most-favoured candidate. It might not seem as equitable as the single transferable vote, the weakness of which Dodgson had already shown in his letters to the *St. James Gazette*.

Moreover, Dodgson's system adhered to axioms (4) and (5). Dodgson discussed candidates' transfer of votes in the context of the party system, but it would have been as reasonable in a no-party race. It nonetheless looks to a modern reader as though Dodgson assumed that an elector who voted Conservative would prefer any Conservative candidate to any Liberal one.

Black (1969) described two other approaches by Dodgson to the problem of further decreasing the percentage of the population unrepresented which he said he found in the Supplement and Postscript to the Supplement of Dodgson (1884g), reprinted in the second edition. One of these (the "conceptual approach") suggests that an equilibrium set of coalitions among re-contracting electors to fill various numbers of seats would produce an optimum set of members, were it not for high transactions costs. The other (the "operational approach") was an attempt by Dodgson (doomed to frustration) to guess on the basis of electoral preference schedules what coalitions they might make.

These approaches do not appear in the Supplement or Postscript to the Supplement. The first ("conceptual") approach cited by Black seems to derive from Thomas Wright Hill's scheme, in which electors form coalitions of Droop quota size and these coalitions select representatives. Black suggested that the "conceptual approach" he mentioned was a predecessor in "what we know today as the coalition games with ordinal utilities" (Black, 1969, 210). Any voting game in which side payments are possible is such a game. The closest Dodgson came to such an approach was to indicate that vote-packing was a way of avoiding "wasted votes", needing the constant supervision of a `caucus' and also a very docile body of Electors, each willing to vote for **any** man on the `right' side-- is a way, but a very clumsy one, for doing this. (Dodgson, 1884e, 5).

Black's "operational approach" is more nearly present in Dodgson's pamphlet and letters. In the Supplement and Postscript, and in the letters of 15 May, 19 May, and 6 June 1884 (1884a, b, and d), Dodgson gave examples of populations with different distributions of preference schedule. He used them, however, to illustrate the aberrations he believed might occur under the voting-systems proposed by the P.R.S., and how his schemes would differ. He did not suggest that he was attempting to divine what coalitions voters would form.

Dodgson's second model, which neither Grofman (1989) nor Black discussed (Black stated this explicitly: 1967, 1), took the numbers of districts, seats, and electors in the country as given, and used the Droop quota Q. Where e is the number of electors in a district and m the numbers of members to be elected by the district, $Q = e/(m+1)$, which is approximately equal to $e/(m+1)$ rounded up to the next integer. Given m, Q is the number of electors each member represents. Dodgson's axiom (1) was that each member should represent approximately the same number of electors, regardless of district. If each district i of n districts has m_i seats and e_i electors, then ideally $(m_i + 1)Q = e_i$ for all i. This yields n equations: by adding them we arrive at $(M + D)Q = E$, where M is the number of members in the House and D the number of districts. On Dodgson's equity of representation principle, then, $Q = E/(M+D)$ and $m_i = e_i(M+D)/E - 1$ (Dodgson, 1884g, 8-10).

Given Q, M, and D, Dodgson compared his method for assigning a district's number of seats with the "rough and ready" method (based on a more standard notion of representation) by which $m = (eM)/E$ with respect to districts of different sizes, given M and D. He found that the rough and ready method assigned too many members to small districts and too few to large ones by his criterion (Dodgson, 1884g, 11-13). He then considered a system based on employing the nation's population P, rather than its number of electors, to compute $Q' = P/(M+D)$. Dodgson gave $m' = e/Q' - 1$, rather than $m' = p/Q' - 1$, where p is the district's

population, but this must be a typo (Dodgson, 1884g, 13-15). Dodgson justified this procedure on the ground as preferable on the eve of the Franchise-Bill, when he wrote, because it would be difficult to estimate the eventual number of electors in each district. He was nonetheless aware that it would give different results unless the proportion of electors to population was nearly the same across districts.

Dodgson next considered what number of votes each elector should be permitted in an m seat race. Dodgson developed a more sophisticated version of the Droop quota which depends on the number of votes each elector can give to m and the number of seats a coalition wishes to fill (Dodgson, 1884g, 16-20). Dodgson used this to draw up a table (Dodgson, 1884g, 21) for the percentage of electors needed to fill s seats out of m, where each elector has v votes to cast to separate candidates, m [1,6], v [1,m] and s [1,m]. This table indicated that where v = m, 51 percent of electors are needed to fill any one seat. As m rises and v approaches 1, it takes fewer electors to fill one seat (or more seats, as long as s is not too close to m). Each entry on the table gave the percentage of the electors who must be in a coalition for the coalition to command s seats, given that each had v votes. The table does not assume that formal coalitions exist, however, and the percentages can be interpreted as the proportion of electors in agreement, whether they know it or not. Dodgson favoured a large number of members per district, with one vote per elector, so that people with minority opinions might nonetheless fill a seat.

Conclusion

While Black (1967, 1) felt that Dodgson's pamphlet was "the most interesting contribution to Political Science that has ever been made", he noted that it had been little noticed and less understood. This is probably due to Dodgson's isolation from the P.R.S. and the methods he used to publish his ideas: letters to the *St. James Gazette*, a pamphlet, and letters to the Tory leader Lord Salisbury, who thought that the method was intended strictly to favour Conservatives, though he didn't mind that. (Dodgson protested that PR had no party bias. Salisbury, however, thought that the extant system in 1884 always favoured the Liberals (Hart, 1992, 75).) The neglect of Dodgson compared with Hare has nothing to do with the acuity of his reasoning, which was well in advance of its time.

Although only Dodgson's pamphlets from the 1870s on voting are at all well known, his later work on voting is of considerable interest. In letters to the *St. James Gazette* and in *Principles of Parliamentary Representation*, Dodgson rigorously analyzed the ideal number of seats per district and votes per elector, and introduced a game-theoretic form of reasoning. Dodgson's method was implicitly one of maximin. Moreover, his analysis includes what we can recognize as a two-stage, sub-game perfect equilibrium. A government attempts to maximize voter welfare according to Dodgson's axioms subject to optimal party behaviour in the second stage. Neglect of many voting schemes associated with proportional representation has created a comparative void in the public choice literature. Recent controversy connected with proportional representation and its aims suggests that public choice theorists might wish to turn to Dodgson's analysis of such systems.

Notes

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1 See Riker's "The Entry of Game Theory into Political Science" in E. R. Weintraub, ed. (1992), *Toward a History of Game Theory*, Annual Supplement to Volume 24, *History of Political Economy*, Durham and London: Duke University Press.

2 Dodgson pioneered game theoretic analysis in public choice by assuming maximizing behaviour of agents. As a rigorous game theoretic model, Dodgson's analysis was preceded only by Waldegrave's 1713 study of maximin play in the card game le Her and Cournot's 1838 treatment of duopoly behaviour. It was more or less contemporary with Bertrand's 1883 review of Cournot.

3 Black (1970, 26) stated that the problem Dodgson considered in PPR was "isomorphic with this, the division in his own nature created by his love for Edith Denman", which, if it amounted to romantic passion, is not evident in Dodgson's published relations with her. Black (1958) similarly remarked that the earlier Dodgson voting pamphlets were his way of expressing turmoil over whether to marry Alice Liddell. Both assertions are sloppy and sketchy. See below for another of Black's curious uses of the term "isomorphic".

4 In the 19th century PR literature in general, raising the cost of party organization to empower the individual voter was an explicit issue (Hart, 1992). Dodgson did not consider costly organization.

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