J.M. Keynes' "Safety First" Approach

Decision Making Under Risk

in the

Treatise on Probability (1921)

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1. Introduction

In 1921, 31 years before A. Roy published his paper on Safety First in *Econometrica*, J.M. Keynes had already presented an equivalent mathematical analysis in Chapters 26 and 29 of his *A Treatise on Probability* (TP). Unfortunately, Chapter 29 was not read and/or understood, possibly due to a very severe typographical error occurring multiple times on four pages of Chapter 29. This paper corrects the typographical error and proves mathematically that Keynes' approach is equivalent to Roy's.

2. Correcting a typographical error

Chebyshev's Inequality, spelled by Keynes as Tchebycheff's, specifies a generalized result which holds for all discrete and continuous random variables having a finite mean and variance. It can be stated in the following manner.

Chebyshev's Theorem: Let X be a random variable having mean μ and variance σ^2, both of which are finite. Then there exists a positive number ε such that

$$P(|X - u| \geq \varepsilon) \leq \sigma^2/\varepsilon^2 .$$

Setting $\varepsilon = k\sigma$, then

$$P(|X - u| \geq k\sigma) \leq 1/k^2$$

Let us now turn to Keynes' exposition. His exposition is not presented in an absolute value notation and this may be the reason it has been overlooked by philosophers and economists. Keynes defines the following two outcome gamble which can be repeated s times where s equals the number of trials. Let A be the amount that is won with a probability of p, where $p = 1 - q$ and $p + q = 1$. Let B be the amount that is lost with a probability of q. Then the probability "that the total winnings (or losses) will lie between the limits

$$s(pA - qB) \pm \propto (A + B)\sqrt{spq} \qquad (1)$$

is greater than $1 - 1/\propto^2$." (Keynes, 1921, p.355; 1973, p.388).[1]

The reader should now note that a typographical error, "a", appears instead of the correct notation, "α". "a" was defined by Keynes "as the mathematical expectation or average value of x". (Keynes, 1921, p.353; 1973, p.386). "α" is correctly defined in the 1973 version as "α is some arbitrary number greater than unity" (Keynes, 1973, p.387, 388) and as "α is small" (Keynes, 1973, p.389). Unfortunately, the typographical error, specifying "a" incorrectly instead of "α",appears a total of 27 separate times on pages 387 to 390 of the 1973 *CWJMK* edition and 30 separate times on pages 354-357 of the original 1921 edition. The symbols that need to be corrected in both editions are:

"$1/\alpha^2$" instead of "$1/a^2$";

"$(\cdot)^2/\alpha^2(\cdot) \geq 1$" instead of "$(\cdot)^2/a^2(\cdot) \geq 1$";

"$1-p < 1/\alpha^2$" instead of "$1-p < 1/a^2$";

"$\mp\alpha\sqrt{\cdot}$" instead of "$\mp a\sqrt{\cdot}$";

"$\alpha = \sqrt{n/t}$" instead of "$a = \sqrt{n/t}$";

"$sp \pm \alpha\sqrt{\cdot}$" instead of "$sp \pm a\sqrt{\cdot}$".

Further, it is recommended that the editors of the 1973 *CWJMK* edition add the following to Keynes' definition that "α is some arbitrary number greater than unity":

"and α represents the number of standard deviations from the mean".

3. Keynes' Analysis in the TP

On pages 355-356 of the 1921 edition (pp.388-389, 1973), Keynes argues that Bernoulli's formula (or binomial distribution) gives answers which are valid only when the number of observations, s, is large, that is for the normal approximation to the binomial. Thus Bernoulli's formula gives exaggerated results. On the other hand, Tchebycheff's formula is "equally valid for all values of s" and has "more cautious limits". Thus, use of Bernoulli's formula is a special case of Tchebycheff's more "general" formula. Keynes correctly argues that the less precise, but more general results of Tchebycheff's inequality are to be preferred to the much more precise, but less general results using Bernoulli's theorem or its normal approximation. One should never just assume that the distribution is normal or binomial. Evidence is required. Let us rewrite equation (1) in a notation more familiar to the average reader. Let $s(pA - qB)$ equal u, the mean or expected return. Let the number of standard deviations $\alpha = k$. Let the random variable $A + B =$ the random variable X. Define $\sqrt{spq} = \sigma =$ the standard deviation. Then we obtain the following result:

$$P(u \pm kX\sigma) \geq 1 - 1/k^2 \tag{2}$$

or

$$1 - P(u \pm kX\sigma) \leq 1/k^2 \tag{3}$$

Let us now make use of absolute value notation. We can rewrite (3) as

$$P(|X - u| \geq k\sigma) \leq 1/k^2 \ .$$

From Chapter 26 of the TP, Keynes defines Risk = R = qpA, where p+q=1 and A is the outcome. Keynes states that R is to be minimized. This is equivalent to minimizing the variance of the outcome A. Let us now combine this result from page 315 (p.348, 1973) of the TP with Keynes' results of his analysis of the gamble using Tchebycheff's inequality.

Another way of writing this inequality is

$$P(|(X-u)/\sigma| > k) \leq 1/k^2 \ . \tag{4}$$

Keynes' rule states that a decision maker is to minimize R. This requires a decision maker to minimize the probability of obtaining an outcome below the mean, or expected value, under the left tail of the distribution. The relevant case that a risk averse decision maker is concerned with is where the lower limit is less than u, where the mean outcome equals $s(pA - qB)$. This means that the term in the absolute value sign will be negative. Thus,

$$P(|(X-u)/\sigma| < -k) \leq 1/k^2 \ . \tag{5}$$

Of course, this result is identical to the lower limit in Roy's Safety First approach, which was defined as the number of standard deviations k lying below the mean or

$$k = (u - u_e)|\sigma , \tag{6}$$

where u_e is the lower limit below which the decision maker does not want his mean return to fall. Roy's rule is to

$$\text{Minimize } P(u < u_e) \ . \tag{7}$$

Since Tchebycheff's inequality holds for all values of k, we obtain

$$P(u < u_e) \leq 1/k^2 \ . \tag{8}$$

The decision maker wants to maximize k. We obtain this result if we invoke Keynes' rule to minimize $R = qpA$ and apply that rule to the problem specified by Keynes in Chapter 29 of the TP.

4. An Application of Keynes' TP Analysis to the long-run purely or perfectly competitive Theory of the firm in the Appendix to Chapter 6 of the General Theory (GT)

On page 68 of the GT, Keynes states that:

> "The concepts of user cost and of supplementary cost also enable us to establish a clearer relationship between long-period supply price and short-period supply price. Long-period cost must obviously include an amount to cover the basic supplementary cost as well as the expected prime cost appropriately averaged over the life of the equipment. That is to say, the long-period cost of the output is equal to the expected sum of the prime cost and the supplementary cost; and, furthermore, in order to yield a normal profit, the long-period supply price must exceed the long-period cost thus calculated by an amount determined by the current rate of interest on loans of comparable term and risk, reckoned as a percentage of the cost of the equipment. Or if we prefer to take a standard "pure" rate of interest, we must include in the long-period cost a third term which we might call the *risk-cost* to cover the unknown possibilities of the actual yield differing from the expected yield. Thus the long

> period supply price is equal to the sum of the prime cost, the supplementary cost, the risk cost, and the interest cost, into which several components it can be analyzed. The short period supply price, on the other hand, is equal to the *marginal* prime cost. The entrepreneur must, therefore, expect, when he buys or constructs his equipment, to cover his supplementary cost, his risk cost, and his interest cost out of the excess of the marginal value of the prime cost over its average value; so that in long-period equilibrium the excess of the marginal prime cost over the average prime cost is equal to the sum of the supplementary, risk, and interest costs.[1]"

We can use Keynes' supply side model of Chapter 20 of the GT, combined with Keynes' strictures to minimize risk from Chapter 26 of the TP, to analyze the above statement. Let $O_r = \varnothing_r(N_r)$ be the r^{th} firm's (industry's) production function. Let Total Variable Cost = TVC = wN_r, where w = the money wage and N_r = the amount of employment in the r^{th} firm or industry, where r can equal 1 or 2. Define F_r = Fixed Cost, which the firm must meet in the long run. Keynes defines p_r = expected price. Let the standard deviation equal σ. Then expected profit, P_r, is equal to

$$P_r = p_r O_r - wN_r - F_r,$$

where $p_r O_r = D_r$ = effective (expected) demand. This can easily be rewritten to conform to Keynes' D-Z analysis, since $Z_r = P_r + wN_r + F_r$ in the long run. D = Z is then specified as

$$D_r = p_r O_r = P_r + wN_r + F_r = Z_r$$

The standard deviation of P_r is σO_r. Using Keynes' rule to minimize risk, we maximize

$$\begin{aligned} k &= \{|p_r O_r - wN_r - F_r|/\sigma O_r\} \\ &= \{|p_r \varnothing(N_r) - wN_r - F_r|\sigma O_r\} \\ &= \{[p_r - (wN_r + F_r)/O_r]\sigma\} \\ &= \{[p_r - ATC_r]/\sigma\} \end{aligned}$$

where ATC_r = average total cost of the r^{th} firm or industry.

Under Keynes' safety first rule, the firm maximizes the difference between the expected price and ATC. This is the same as minimizing ATC. This minimizes the probability of the firm having to shut down in the long run. Since the level of output produced under risk aversion is less than that produced using the standard perfect competition rule P = MR = MC, the level of employment will also be smaller. Thus, reducing risk will increase the level of employment. The same result holds in the aggregate if we sum over all r firms and r = 2 industries to obtain aggregated employment $N = \sum_r N_r = N_1 + N_2$

5. Conclusions

If Keynes' results in Chapter 29 of the TP are combined with Keynes' rule to minimize R = pqA or R = npqA, one obtains the same analytic result published by A. Roy some 31 years after Keynes published his TP in 1921.

Keynes' discussion of Tchebycheff's Inequality, within the context of the decision theoretic lottery problem of Chapter 29 of the TP, allows one to tie Keynes' long run analysis of a perfectly (purely) competitive firm, in his appendix to Chapter 6, to the short run analysis of such a firm in Chapter 20 of the GT.

Finally, this paper corrects a very severe and extensive typographical error which may account for the chapter having been overlooked for 75 years.

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Note

1. In my opinion, Paul Samuelson (1977) presents an unsatisfactory analysis of Keynes' Theory of Risk. First, he states that Keynes is "proposing a new (and dubious) concept of risk." (Samuelson, 1977, p.46). Samuelson is only half correct; it is new, but it certainly is not dubious, unless Samuelson considers Roy's principle dubious.

Samuelson appears to have looked in the index to the TP and found the subject risk listed on page 315 (1921). From this he concludes, erroneously, that "Keynes [sic] words on risk are so brief they can essentially be reproduced in full". (Samuelson, p.46). He then cites from pages 314-315 (1921) of the TP.

Next, Samuelson claims, again erroneously, that

> "Since Keynes has not spelled out exactly how he would define his "risk" in the general situation of several algebraic gains and losses with their respective defined probabilities $\left(A_1,\ldots,A_n;p_1,\ldots,p_n=1-\sum_1^{n-1}p_j\right)$, it will be best to stay with his case of a single monetary gain A, that will obtain with probability p. Thus, we consider $(A_1,A_2;p_1,p_2)\equiv(A,0;p,1-p)$".

Of course, Keynes spelled out exactly what his definition would be on page 355 of the TP (1921). Setting s = 1 and B = 0, we obtain, from $s(pA-qB)$, $1(pA-q0)=pA$. Keynes' example from Chapter 26 is a simplified version of the problem in Chapter 29. I believe Samuelson overlooked Chapter 29 of the TP.

Finally, Keynes gives an extensive discussion of Bernoulli's Theorem (Binomial Distribution) in Chapter 29 on pages 338-352 (1921), including the normal approximation to the Binomial when the number of observations, m, becomes large (pages 338-339) by means of the use of Stirling's Theorem. Keynes' simplified example is binomial (Bernoulli) with only one trial. Keynes' example is more than sufficient to get his point across. Thus, the mean value is

$$E[X]=\sum_{x=0}^{1}Xf(X),$$

where X is a random variable. "Since each of the variables X_i assumes the values 0 and 1 with the probability q and p, respectively, it follows that $u_i=E[X_i]=0\cdot q+1\cdot p=p$". (Hoel, 1971, p.120; also see Goldman, 1970, pp.187-188). Then $u_w = np$ for a sequence of n Bernoulli trials $W = X_1 + \ldots + X_n$. Further,

$$
\begin{aligned}
E[X_i - u_i]^2 &= \sigma^2 = E[X_i - p]^2 \\
&= (0-p)^2 q + (1-p)^2 p \\
&= p^2 q + q^2 p \\
&= p(pq) + q(qp) \\
&= (p+q)(pq) \\
&= pq.
\end{aligned}
$$

Thus, $\sigma_w^2 = npq$. For n = 1, we obtain pq, the variance for 1 trial. Keynes specifies $\sigma_w^2 = mpq$ on page 339 of the TP (1921), where now, instead of s, m = number of trials, p = probability of success, q = probability of failure, and $mpq = \sigma^2$. Keynes then incorporates the variance in his generalization of Bernoulli's formula (binomial formula) to the Normal Distribution, i.e.,

$$1/\sqrt{2\pi mpq}\int_{-a}^{+a} e^{-z^2}/2mpq.\,dz\ .$$

Samuelson's conclusion, that "For more than half a century, nothing useful seems to have come from Keynes' 1921 innovation of his new 'risk' concept", is unfortunately true. However, Keynes can't be blamed for the failure of his readers to comprehend his analysis, which is mathematically correct, in Chapter 29 of the TP.

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